

Problems and Results in Graph Theory

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ABSTRACT

I published many papers on this and related subjects. This paper will contain relatively little new material. I just give a short discussion of some problems which I thought about in the last few months.

1. Ramsey's theorem and generalizations.

First I discuss some problems connected with Ramsey's theorem and its generalizations. $r(G_1, G_2)$ is the smallest integer ℓ so that if we color the edges of K_ℓ by two colors in an arbitrary way, then there always is either a monochromatic G_1 in the first color or a monochromatic G_2 in the second color. Burr has two excellent survey articles on this subject. If G_1 is a K_m and G_2 a K_n we write $r(G_1, G_2)$ as $r(m, n)$. It is well known that

$$c_1 n 2^{n/2} < r(n, n) < c_2 \binom{2n-2}{n-1} \quad (c_2 < 1). \quad (1)$$

In several of my older results I stated that

$$r(n, n) < c_3 \binom{2n}{n} (\log \log n)^{-1}, \quad \text{but recently J. Winn and E.}$$

Levine threw some doubts on the validity of the proof of this result. I offer 100 dollars for the proof that

$$\lim_{n \rightarrow \infty} r(n, n)^{1/n} = C \quad (2)$$

exists and another 250 dollars for the value of C . I also offer 100 dollars for a constructive proof of

$$r(n,n) > (1+c)^n. \quad (3)$$

Peter Frankl gave a nonconstructive proof of $r(n,n) > n^k$ for every k if $n > n_0(k)$ and this was improved by F. Chung to

$$r(n,n) > e^{c(\log n)^{4/3}},$$

and Peter Frankl proved $r(n,n) > e^{c(\log n)^2}$, but so far a constructive proof of (3) is nowhere in sight.

We have

$$\frac{c_2 n^2}{(\log n)^2} < r(3,n) < \frac{c_1 n^2}{\log n}. \quad (4)$$

The lower bound of (4) was proved by me, the upper bound by Ajtai, Komlós and Szemerédi in a forthcoming paper which will be published in the European Journal of Combinatorics. Graver and Yackel proved more than 10 years ago that $r(3,n) < \frac{cn^2 \log \log n}{\log n}$.

Both the upper and the lower bound of (4) are proved by probabilistic methods. The probabilistic method very likely eventually will give

$$r(k,n) > n^{k-1-\epsilon}, \quad (5)$$

but so far the proof of (5) even for $k = 4$ seems to present great difficulties.

There seems to be no doubt that

$$\lim_{n \rightarrow \infty} r(n+1, n+1)/r(n, n) = C$$

and that

$$\lim_{n \rightarrow \infty} r(n+1, n)/r(n, n) = C^{1/2},$$

but we could not even prove that

$$r(n+1, n) - r(n, n) > n^k \quad (6)$$

holds for every k if $n > n_0(k)$. We could prove (6) only for $k = 2$; also $r(n+1, 3) - r(n, 3) \rightarrow \infty$ seems to present dif-

ficulties. "We" in this case stands for Burr, Faudree, Rousseau, Schelp, V.T. Sós and myself. For references, see [3, 4, 5, 12, 13, 14, 15].

2. The size Ramsey number.

In a quadruple paper Faudree, Rousseau, Schelp and I started to investigate the size Ramsey number $\hat{r}(G_1, G_2)$, the smallest integer for which there is a graph G of $\hat{r}(G_1, G_2)$ edges so that if we color the edges of G by two colors either color I contains a copy of G_1 or color II a copy of G_2 . Our most interesting unsolved problem states: Denote by P_n a path of length n , and write $\hat{r}(P_n)$ instead of $\hat{r}(P_n, P_n)$. Is it true that

$$\hat{r}(P_n)/n \rightarrow \infty, \hat{r}(P_n)/n^2 \rightarrow 0? \tag{7}$$

We are sure that the first equation of (7) holds, but are less sure about the second.

A few days ago we started to investigate $\hat{r}(K(1, n), K(m))$ where $K(u, v)$ denotes a complete bipartite graph of u white and v black vertices. We are reasonably sure that we soon will have a simple explicit formula for $\hat{r}(K(1, n), K(m))$ for every n and m . One of the lemmas we will need states that if G has $\binom{2n+1}{2} - \binom{n}{2} - 1$ edges, then G is the union of a bipartite graph and a graph each vertex of which has degree $< n$. Faudree has a very simple proof if G has at most $2n + 1$ vertices and he has an induction proof which surely will give the general case. To compute $\hat{r}(K(1, n), K(m))$ we will have to generalize the lemma where a bipartite graph is replaced by an r -partite graph but we do not expect serious difficulties. We also tried to determine $\hat{r}(K(2, n), K(m))$, but here we are much less optimistic. We conjectured that for every ℓ and $n > n_0(\ell)$

$$\hat{r}(K(\ell, n), K(\ell, n)) = 2(n-1) \binom{2\ell-1}{\ell} + 2\ell - 1. \tag{8}$$

Equation (8) is trivial for $\ell = 1$, but we could not prove it for $\ell > 1$. We hope to return to a more detailed study of the size Ramsey numbers in forthcoming publications. For reference, see [7].

3. Some Problems on Extremal Graph Theory.

Recently an excellent book of B. Bollobás appeared on this topic. Also Simonovits and I have several papers on this topic (many of them are joint papers). Here I just mention a few striking unsolved problems.

Let $G(k;\ell)$ be a graph of k vertices and ℓ edges. $f(n;G(k;\ell))$ is the smallest integer for which every $G(n;f(n;G(k;\ell)))$ contains our $G(k,\ell)$ as a subgraph. $f(n;k,\ell)$ is the smallest integer for which every $G(n;f(n;k,\ell))$ contains at least one $G(k;\ell)$ as a subgraph. In this note we will only consider bipartite graphs $G(k;\ell)$. By the well known results of Stone, Simonovits and myself the asymptotic behavior of $f(n;G(k;\ell))$ is determined by the chromatic number of $G(k;\ell)$. Here we assume that $G(k;\ell)$ is bipartite.

Brown, V.T. Sós, Rényi and I proved that

$$f(n;C_4) = \left(\frac{1}{2} + o(1)\right)n^{3/2}. \quad (9)$$

In previous papers I conjectured

$$f(n;C_4) = \frac{1}{2}n^{3/2} + \left(\frac{1}{4} + o(1)\right)\frac{n}{4}, \quad (10)$$

but we are very far from being able to prove (10).

Let p be a prime or a power of a prime. If $n = p^2 + p + 1$, we conjectured that

$$f(n;C_4) = p^2(p+1) + p(p+1)/2 + 1. \quad (11)$$

Equation (11) has very recently been proved by Füredi if $p = 2^k$. The general case is still open. Füredi's paper will be published soon.

If we assume that our $G(n;t)$ has no C_4 and C_3 then probably

$$\max t = \left(\frac{1}{2\sqrt{2}} + o(1) \right) n^{3/2} \tag{12}$$

and probably $G(n;t)$ can be assumed to be bipartite.

Simonovits and I proved this if we assume that $G(n;t)$ does not contain a C_4 and contains no odd circuit of length ≤ 11 .

Our results are not yet published.

Let $C^{(n)}$ be the graph of the n -dimensional cube. $C^{(n)}$ has 2^n vertices and $n2^{n-1}$ edges, and $C^{(2)}$ is of course a C_4 . Simonovits and I proved that

$$f(n;C^{(3)}) < c_1 n^{8/5}$$

but $f(n;C^{(3)} - e) < c_2 n^{3/2}$. We conjectured that for every

bipartite $G(k;l)$, where $l \geq k$, there is a rational

$\alpha_{k,l}$, where $1 < \alpha_{k,l} < 2$, for which

$$f(n;G(k;l))/n^{\alpha_{k,l}} \rightarrow c, \quad 0 < c < \infty, \tag{13}$$

and conversely for every rational α , with $1 < \alpha < 2$ there is a bipartite $G(k;l)$ for which (13) holds. We are very far from being able to prove this conjecture.

We further conjectured that if $G(k;l)$ is a bipartite graph each vertex of which has valency (or degree) ≥ 3 then there is an $\epsilon_k > 0$ so that

$$f(n;G(k,l)) > n^{\frac{3}{2} + \epsilon_k}. \tag{14}$$

Conversely we conjectured that if $G(k;l)$ is such that every subgraph has a vertex of degree ≤ 2 then

$$f(n;G(k;l)) < c_k n^{3/2}. \tag{15}$$

We are very far from being able to prove (14) or (15), which seem to me to be very attractive conjectures. We further posed the following problems. Assume that (13) holds. What are the possible values for $\alpha_{k,l}$ for all the possible bipartite

graphs satisfying $k \leq l \leq \left\lfloor \frac{k^2}{4} \right\rfloor$? What is the largest possible $r = r_k$ for which there is a bipartite $G(K,l)$ so that for all

possible choices of the edges e_1, \dots, e_r the bipartite graph $G(k; \ell + e_1 + \dots + e_r) = G(k; \ell + r)$ has the same α as $G(k; \ell)$? Perhaps $r_{2k} = 2k - 1$. Let $\ell = (k-1)^2 + 1$. $G(2k; \ell)$ has the vertices $x_1, \dots, x_k, y_1, \dots, y_k$, its edges are (x_i, y_j) , $1 \leq i \leq k-1, 1 \leq j \leq k-1$ and (x_k, y_k) . It has often been conjectured that $K(k-1, k-1)$ has $\alpha = 2 - \frac{1}{k-1}$. An old result of Kövari, V.T. Sós, Turán and myself states that $\alpha \leq 2 - \frac{1}{k-1}$ for this graph and an old result of myself states that the α belonging to $K(k, k) - e$ is also $\leq 2 - \frac{1}{k-1}$ and we obtain $K(k, k) - e$ from our graph by adding $2k - 1$ edges.

What is the largest R_k for which there is a $G(k; \ell)$ so that the omission of any R_k edges does not decrease α ?

Finally, I state an old and interesting conjecture of V.T. Sós and myself. Is it true that every $G(n; \lfloor \frac{n(k-1)}{2} \rfloor + 1)$ contains as a subgraph every tree of $k+1$ vertices? The conjecture, if true, is best possible since a $G(n; \lfloor \frac{n(k-1)}{2} \rfloor)$ does not have to contain a star $K(1, k)$. Szemerédi has a recent result which seems to prove a slightly weaker result. For references, see [2, 10].

4. Some Applications of the probability method.

I conjectured and Ajtai, Komlós and Szemerédi proved that there is an $f(c)$, $f(c) > 0$ for $c > \frac{1}{2}$ and $f(c) \rightarrow 1$ as $c \rightarrow \infty$, so that every $G(n; cn)$ contains a path of length $(f(c) + o(1))n$. In their forthcoming paper they strengthened and extended several further results of a paper of Rényi and myself.

Ajtai, Komlós and Szemerédi proved the following surprising and interesting result. Let $G(n; [kn])$ be a graph which contains no triangle. Then it contains an independent set having more than $\frac{cn \log k}{k}$ vertices. Without the factor $\log k$ the result is trivial and the course does not need the assumption

that $G(n;[kn])$ contains no triangle. They also show that apart from the value of c their result is best possible. It seems likely that for every r there is a $g_r(k)$ which tends to infinity as k tends to infinity, so that if $G(n;[kn])$ is a graph which contains no $K(r)$ then it contains an independent set of more than $cn g_r(k)/k$ vertices. But this conjecture is open for every $r \geq 4$.

Some time ago I heard the following conjecture. Let $G(n)$ be a graph of n vertices. Denote by $C(G(n))$ the smallest integer for which the vertices of $G(n)$ can be covered by $C(G(n))$ cliques. $E(G(n))$ is the largest integer for which $G(n)$ has $E(G(n))$ edges no two of which are in the same clique. Is it true that $E(G(n)) = C(G(n))$? The probability method easily gives that the conjecture is false. If we choose each edge of $G(n)$ with probability $\frac{1}{2}$ then for almost all graphs $E(G(n)) < c_1 (\log n)^2$, and it is well known and easy to see that the largest clique contained in almost all graphs is less than $c_2 \log n$. Thus, for almost all graphs,

$$\frac{C(G(n))}{E(G(n))} > c \frac{n}{(\log n)^3} .$$

In fact it is not hard to prove that

$$c_4 \frac{n}{(\log n)^3} < \max_{G(n)} \frac{C(G(n))}{E(G(n))} < c_3 \frac{n}{(\log n)^3} .$$

Let t be fixed and sufficiently large. The probability method gives that there is a $G(n)$ with $E(G(n)) \leq t$ and $C(G(n)) > n^{c_t}$. I did not succeed in determining the smallest value of t for which this holds for some $C_t > 0$. It would be easy to give crude upper bounds for t .

Rényi and I proved that almost all graphs $G(2n; (1+\epsilon)n \log n)$ have a complete matching - the result is no longer true without the additive term ϵ . In fact we proved a

stronger result. Recently Shamir asked the following very nice question. For $|G| = 3n$, how many triples of G must we choose so that for almost all choices of the triples there should be a subsystem of n disjoint triples? Shamir showed that n^2 triples certainly suffice. But the correct order of magnitude may be very much less. For references, see [8, 9, 16, 18].

5. Miscellaneous problems.

Hajnal and I conjectured that if G has infinite chromatic number then $\sum \frac{1}{2r+1} = \infty$, where $2r+1$ runs through the integers for which G has a circuit of length $2r+1$. In fact perhaps the set of these $(2r+1)$'s has positive upper density. As far as I know, nothing has been done with this conjecture. We also conjectured that for every $G(n;kn)$, $\sum \frac{1}{r} > c \log k$, where r runs through the integers for which our $G(n;kn)$ has a circuit of length r . It is easy to see that this conjecture, if true, is best possible. Szemerédi and Gyárfas very recently proved this conjecture; probably his method will give the best possible value for c .

During our meeting I heard from mathematicians at New Orleans the following nice problem. Let $G(n)$ be a graph of n vertices. ($G(n)$ is not complete and not the empty graph.) Assume that if we add any new edge to $G(n)$ (i.e., we join two vertices of $G(n)$ which are not joined in $G(n)$) then the clique number always increases and further if we omit an arbitrary edge of $G(n)$ then the independence number increases. Can one characterize these graphs? Are there infinitely many of them? So far, only three such graphs are known: C_5 ; the unique graph of 17 vertices which, together with its complement, has no K_4 ; and, as V.T. Sós observed, the graph of 13 vertices which has no K_3 and whose complement has no K_5 .

Let E be the n -dimensional Euclidean space. Join two points of E_n if their distance is 1. Denote by $\chi(E_n)$ the chromatic number of this graph. These problems were initiated by Hadwiger and Nelson. It was conjectured that $\chi(E_2) = 4$, but now it is generally believed that $\chi(E_2) > 4$. In any case it is known that $\chi(E_2) \leq 7$.

I conjectured that if $|S| = n$ then to every $\epsilon > 0$ there is an n so that, if

$$A_i \subset S, 1 \leq i \leq t_n, t_n \geq (2 - \epsilon)^n,$$

then for every $\eta n < r < (\frac{1}{2} - \eta)n$ there are two sets A_i and A_j for which $|A_i \cap A_j| = r$. This conjecture, if true, would imply that $\chi(E_n) > (1+c)^n$. Very recently Peter Frankl proved a slightly weaker result from which he immediately deduced $\chi(E_n) > (1+c)^n$. Larman and Rogers proved $\chi(E_n) < (3+o(1))^n$; presumably

$$\lim_{n \rightarrow \infty} \chi(E_n)^{1/n} = C$$

exists and it would be interesting to determine C .

In a forthcoming paper, Simonovits and I investigate (among others) the following problem. Join two points of the n -dimensional unit sphere by an edge if their distance is 1. Determine or estimate the chromatic number of this graph. Trivially it is greater than cn , but perhaps it tends to infinity exponentially.

I asked: Let S be a set in E_2 . Join two points of S if their distance is 1. Assume that the girth of this graph is k . Is it true that for every k there is a set S so that the resulting graph has chromatic number ≥ 4 ? Wormald proved that such a graph exists for $k = 5$; for $k > 5$ the problem is open. A well known theorem of de Bruijn and myself states that every infinite graph of chromatic number k contains a finite subgraph of chromatic number k , thus all these

problems are really problems on finite graphs. For example, if $\chi(E_2) > 4$, then there is a finite set of points so that if we join two points where distance is 1 we get a graph of chromatic number > 4 . The only trouble is that no estimate is available on the size of this set. One final problem about these graphs: Denote by $f_2(k)$ the largest integer for which there are k numbers $\alpha_1, \dots, \alpha_k$ so that, if we join two points of the plane when their distance is one of the α_i , $i = 1, \dots, k$, then the chromatic number of this graph is $f_2(k)$. Estimate $f_2(k)$ as well as you can. Is $f_2(k)$ of polynomial or exponential growth? Or is it something in between?

Silverman and I considered the following question: Let G be a graph whose vertices are the integers. Join i and j if the sum $i + j$ is a square. Is it true that $\chi(G) = \aleph_0$? Clearly many related problems can be asked. For references, see [1, 11, 17, 19].

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