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ON GRAPHS WHICH CONTAIN ALL SPARSE GRAPHS

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Dedicated to Professor A. Kotzig on the occasion of his sixtieth birthday

1. Introduction

Let $\mathscr{H}_n$ denote the class of all graphs with n edges and denote by $s(n)$ the minimum number of edges a graph G can have which contains all $H \in \mathscr{H}_n$ as subgraphs. In this paper we establish the following bounds on $s(n)$:

Theorem 1.

$$\frac{cn^2}{\log^2 n} < s(n) < \frac{c'n^2 \log\log n}{\log n}$$

for n sufficiently large and c, c' some constants.

We also consider the problem of determining the minimum number of edges, denoted by $s'(n)$, a graph can have which contains every planar graph on n edges as a subgraph. We prove:

Theorem 2. $s'(n) < cn^{3/2}$.

In [1, 2, 3], two of the authors investigated the problem of determining the minimum number of edges a graph or a tree could have which contains all trees on n edges as subgraphs. For a brief survey on these 'universal' graphs the reader is referred to [4].

2. A lower bound for $s(n)$

Let G be a graph which contains all graphs on n edges. Suppose G has t edges. Thus G contains at most $\binom{t}{n}$ different subgraphs on n edges.

On the other hand, G contains all graphs on n edges and $\lfloor n/\log n \rfloor$ vertices where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. There are at least

$$\binom{\binom{\lfloor n/\log n\rfloor}{2}}{n} \cdot \frac{1}{\lfloor n/\log n\rfloor !}$$

different graphs with n edges and $\lfloor n/\log n \rfloor$ vertices (see [5]). Therefore we have

$$\binom{t}{n} \geqslant \binom{\binom{\lfloor n/\log n\rfloor}{2}}{n} \cdot \frac{1}{\lfloor n/\log n\rfloor !}.$$

By a straightforward calculation, this implies

$$t \geqslant cn^2/\log^2 n$$

for some constant c.

Hence we have shown $s(n) > cn^2/\log^2 n$.

3. An upper bound for $s(n)$

We will prove (by the probability method) that there exists a graph with $cn^2 \log\log n/\log n$ edges[1] that contains all graphs with at most n edges. The existence of such a graph will follow from the following sequence of observations.

Claim 1. *Given positive integers a and b where $a < b < a \log a$ and $\log\log\log a \geqslant 1$, there is a bipartite graph H with vertex set $A \cup B$ where $|A| = a$ and $|B| = b$, which satisfies the following conditions:*

(i) *H has no more than abp edges where $p = \log\log a/\log a$;*

(ii) *For any k disjoint subsets of B, say, $S_1, \ldots, S_k$, each with cardinality at most p^{-1}, and $2kp^{-2} < a$, we have*

$$\left|\bigcup_{i=1}^{k} N(S_i)\right| \geqslant kp^{-2}$$

where

$$N(S_i) \equiv \{v \in A : v \text{ is adjacent to all vertices in } S_i\}.$$

Proof. We consider the set of all bipartite graphs on a and b vertices with abp edges. For a set $S_i \subset B$, $|s_i| < d = p^{-1}$, the probability of a vertex v in A being in $N(S_i)$, is at least p^d. Therefore the probability of v not being in any $N(S_i)$ is at most $(1-p^d)^k$. The

[1] Strictly speaking, we should use $3n[\log\log n/\log n]$ or $[3n \log\log n/\log n]$ since $|A|$ is an integer. However, we will usually not bother with this type of detail since it has no significant effect on the arguments or results.

probability that there are $a-kd^2$ vertices in A not in any $N(S_i)$ is at most

$$\binom{a}{kd^2}(1-p^d)^{k(a-kd^2)} \leqslant 2^a e^{-p^d ka/2}.$$

Since there are at most b^{dk} choices for S_i, $1 \leqslant i \leqslant k$, the probability for a bipartite graph to be 'bad' is at most

$$b^{dk} \cdot 2^a e^{-p^d ka/2} < (a \log a)^{p^{-1}k} \cdot 2^a e^{-p^d ka/2}$$
$$< (a \log a)^{a \log\log a/\log a} 2^a e^{-a^2 p^{d-2}/4} < 1'$$

Therefore the required bipartite graph exists as claimed.

Claim 2. *Given positive integers a and b where* $a < b < a \log a$ *and* $\log\log\log a \geqslant 1$, *there is a bipartite graph H with vertex set* $A \cup B$ *where* $|A| = a$ *and* $|B| = b$ *satisfying the following conditions:*

(i) *H has no more than abp edges where* $p = \log\log a/\log a$.

(ii) *Let* H' *be a bipartite graph with vertex set* $X \cup Y$ *where* $|X| \leqslant \frac{1}{2}a$, $|Y| = b$ *and maximum degree* p^{-1}. *Then* H' *can be embedded in H in the* strong *sense, i.e. any one-to-one map* $\lambda: Y \to B$ *can be extended to* $\bar{\lambda}: X \cup Y \to A \cup B$ *such that* $\bar{\lambda}(u)$ *is adjacent to* $\bar{\lambda}(v)$ *in H if u is adjacent to v in* H'.

Proof. We take H to be the graph in Claim 1. The mapping λ will be extended to $\bar{\lambda}: X \cup Y \to A \cup B$ in the following way:

For a vertex x in X, we define

$$S(x) = \{b \in B : b = \lambda(y) \text{ and } y \text{ is adjacent to } x\},$$

$$M(x) = N(S(x)) = \{v \in A : v \text{ is adjacent to all vertices in } S(x)\}.$$

The existence of $\bar{\lambda}$ is equivalent to a system of distinct representatives for $\{M(x)\}_{x \in X}$.

It suffices to show that for any set $X' \subseteq X$ we have

$$\left|\bigcup_{x \in X'} M(x)\right| \geqslant |X'|.$$

This is clearly true for $|X'| \leqslant (\log a/\log\log a)^2$ by property (ii) of H.

Now suppose $|X'| > (\log a/\log\log a)^2$. Since H' is of bounded degree $d = \log a/\log\log a$, for each x there are at most d^2 vertices x' in X with $S(x) \cap S(x') \neq \emptyset$. Thus there is a subset X'' of X where $|X''| \geqslant |X'|/d^2$ such that all $S(x)$, $x \in X''$, are mutually disjoint. Therefore,

$$\left|\bigcup_{x \in X'} M(x)\right| \geqslant \left|\bigcup_{x \in X''} M(x)\right| \geqslant \frac{|X'|p^{-2}}{d^2} \geqslant |X'|.$$

This completes the proof of Claim 2.

Claim 3. *There exists a graph* $\bar{H}$ *with* $4n^2 \log\log n/\log n$ *edges which contains all graphs with n vertices and degree at most* $\log n/\log\log n = d$.

Proof. We will construct a d-partite graph $\bar{H}$ as follows:

(i) $\bar{H}$ has vertex set $A_1 \cup A_2 \cup \cdots \cup A_{d+1}$ with $|A_i| = 2n/d$ for each i;

(ii) For each i, no $u, v \in A_i$ are adjacent;

(iii) The edges between A_i and $A_1 \cup A_2 \cup \cdots \cup A_{i-1}$ form a graph described in Claim 2.

It can be easily seen that $\bar{H}$ has at most $4n^2 \log\log n/\log n$ edges. It suffices to prove that any graph G with degree d can be embedded in $\bar{H}$. A nice result of Hajnal and Szemerédi [6] states that any graph with degree at most d can be colored by $d+1$ colors in such a way that the sizes of the color classes differ by at most 1. Suppose G has color classes $C_1, \ldots, C_{d+1}$. We will then embed C_1 into A_1, C_2 into A_2, and so on, as guaranteed by Claim 2.

Claim 4. *There exists a graph $F(n)$ with $Cn^2 \log\log n \log n$ edges which contains all graphs on n edges where C is an absolute constant.*

Proof. We will construct the graph $F(n)$ as follows:

(i) The vertex set is the disjoint union of A and B where $|A| = 2n \log\log n/\log n$ and $|B| = 2n$.

(ii) Every vertex v in A is adjacent to all vertices in $V(F(n)) - \{v\}$.

(iii) The subgraph of $F(n)$ induced by B is the graph, as described in Claim 3, which has $4n^2 \log\log n/\log n$ edges and contains all graphs with $2n$ vertices and degree at most d.

It is easy to see that $F(n)$ has at most $10n^2 \log\log n/n^2$ edges. Let G be an arbitrary graph on n edges. G has at most $2n \log\log n/\log n$ vertices with degree more than $\log n/\log\log n$. These vertices will be embedded in A. The remaining part of the graph will then be embedded in B as guaranteed by Claim 3.

This completes the proof of Claim 4.

Remark. If instead of using the result of Hajnal and Szemerédi, we use the simple fact that a graph on n vertices and maximum degree d can be $2(d+1)$ colored so that each color class has size at most n/d, then the resulting bound will differ from the one presented by a constant factor.

4. Universal graphs for planar graphs

We will use the following theorem to give an upper bound of $n^{3/2}$ for the universal graphs which contain all planar graphs on n edges.

Separator Theorem (Lipton and Tarjan [6]). *Let G be any planar graph with n vertices. The vertices of G can be partitioned into three sets, A, B, C such that no edge joins a vertex in B with a vertex in C, neither B and C contain more than $n/2$ vertices, and A contains no more than $2\sqrt{2n}/(1-\sqrt{2/3})$ vertices.*

Let $G(m)$ denote the graph constructed as shown in Fig. 1.

The vertices of $G(n)$ can be partitioned into three parts, X, Y and Z where $|X| =$

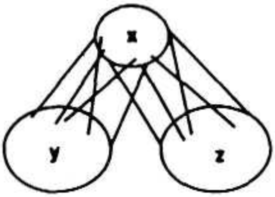

Fig. 1.

$2\sqrt{2n}/(1-\sqrt{2/3})=c_1\sqrt{n}$, $|Y|=|V(G(\lfloor n/2\rfloor))|$ and $|Z|=|V(G(\lfloor n/2\rfloor))|$. Any vertex in X is adjacent to any vertex in $G(n)$ except itself. The induced subgraph on Y is $G(\lfloor n/2\rfloor)$ and the induced subgraph on Z is $G(\lfloor n/2\rfloor)$.

It is rather straightforward to see that any planar graph with n vertices can be embedded in $G(n)$ since we can partition any planar graph into three parts, A, B and C as described in the Separator Theorem, and we can embed A in X, B in Y and C in Z.

We also note that $G(n)$ has fewer than $c_2 n$ vertices since

$$|V(G(n))|<2|V(G(n/2))|+c_1\sqrt{n}$$

and we can prove by induction on n that

$$|V(G(n))|\leqslant\frac{c_1\sqrt{2}}{\sqrt{2}-1}n\left(1-\frac{1}{\sqrt{2n}}\right).$$

Now, by the construction of $G(n)$, we know that

$$|E(G(n))|\leqslant|V(G(n))|\cdot c_1\sqrt{n}+2|E(G(n/2))|.$$

It follows by induction that $G(n)$ has fewer than $cn^{3/2}$ edges where $c=c_1^2\sqrt{2}/(\sqrt{2}-1)=19.7607\ldots$. Therefore we have

$$s'(n)<cn^{3/2}$$

and Theorem 2 is proved.

We note that the obvious lower bound for $s'(n)$ is $\frac{1}{2}n\log n$ which is the lower bound for the number of edges in graphs which contains all trees on n edges (see [2]). At present we do not know any better lower bound than $cn\log n$.

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