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SOME OF MY FAVOURITE PROBLEMS WHICH RECENTLY HAVE BEEN SOLVED

Paul Erdös

Hungarian Academy of Science
Budapest
Hungary

During my very long mathematical life, which has now extended for 50 years, I made conjectures in different subjects. My conjectures in set theory most of which are joint work with Hajnal, Rado, Milner and others I shall practically ignore since several survey papers appeared on this subject - a new one would clearly be needed but I am certainly not competent to write it alone. Thus I will concentrate mainly on number theory, geometry and various branches of analysis and also just for completeness I mention some of my few conjectures in topology.[1] I will almost entirely discuss only solved or at least partially solved conjectures and in order not to make the paper too long I do not attempt completeness, also since I have to finish the paper in somewhat of a hurry, I shall have to rely greatly on my memory which despite my enormous age, is still quite good but is not (and in-fact never was) infallible and so I apologise for the omissions.

First of all I state some of my oldest conjectures all from the early thirties (1930's not 1830's).

1. Let $1 \leq a_1 < \dots < a_k \leq n$ be a sequence of integers. Let $\varepsilon_i = 0$ or 1 and assume that all the 2^k sums $\Sigma_{i=1}^{k} \varepsilon_i a_i$ are all distinct. Is it true that

(1) $$\max k = \frac{\log n}{\log 2} + o(1) \text{ ?}$$

and is it true that for some n

(2) $$\max k > \left[\frac{\log n}{\log 2}\right] + 1 \text{ ?}$$

This conjecture was published only in the late fifties and was before this date rediscovered by L. Moser. (1) is still open, the best upper bound known for (1), due to Moser and myself, is

$$\text{Max } k < \frac{\log n}{\log 2} + \frac{\log \log n}{2 \log 2} + o(1).$$

(2) was proved by Conway and Guy and it has been conjectured by some that in fact $\max k \leq \left[\frac{\log n}{\log 2}\right] + 2$. It has not yet been proved that (2) holds for all suffi-

[1] Since I wrote several survey papers on number theory and combinatorics, I will mention mainly relatively recently settled conjectures.

ciently large n.

P. Erdös, Problems and results in additive number theory, Coll. sur la théorie des nombres, Bruxelles, George Thone, Liège; Masson et Lie, Paris, 1955 127-137.

J. H. Conway and R. K. Guy, Solution of a problem of P. Erdös, Colloq. Math. 20 (1969), 307.

2. Let $f(n) = \pm 1$ be an arbitrary function defined on the integers. Is it true that to every C there is a d and an m so that

$$(1) \qquad \left| \sum_{k=1}^{m} f(kd) \right| > C \ ?$$

This conjecture which was made of course under the influence of the (then new) theorem of van der Waerden on arithmetic progressions has never been seriously attacked.

A weaker form of (1) states : If $f(n) = \pm 1$ is completely multiplicative, then $\left| \sum_{k=1}^{n} f(k) \right|$ is unbounded.

3. Finally here is my old conjecture with Turán which was also made under the influence of van der Waerden's conjecture : Is it true that every sequence of positive density contains arbitrarily long arithmetic progressions ?

This conjecture was proved nearly 10 years ago by Szemerédi in a most ingenious way. Later a quite novel proof using ergodic theory was found by Fürstenberg. I discussed this problem in several survey papers thus here I restrict myself to stating a stronger conjecture of mine : Is it true that if $1 \leq a_1 < \dots$ is an infinite sequence of integers for which $\sum \frac{1}{a_i} = \infty$ then the a's contain arbitrarily long arithmetic progressions ? I offer 3000 U.S. dollars for a proof (or disproof) of this conjecture. If true, this would of course, imply that there are arbitrarily long arithmetic progressions among the primes.

E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, Acta. Arith. 27 (1975), 199-245. This paper contains extensive references to the older literature.

H. Fürstenberg, Ergodic behaviour of diagonal measures and a theorem of Szemerédi, J. d'Analyse Math. 31 (1977) 209 - 256.

H. Fürstenberg and Y. Katznelson, An ergodic Szemerédi theorem for commuting transformations, J. Analyse Math. 34 (1978), 275-291.

For many further problems and results on combinatorial number theory see my booklet with R. L. Graham, Old and new problems and results in combinatorial number theory, Monographie No. 28 de L'Enseiguement Math., 1981.

I

In this chapter I discuss conjectures in geometry.

§1. I have to start with a prehistoric conjecture, the so called Erdős-Mordell inequality : In 1932 I conjectured that if ABC is a triangle and 0 a point in the interior, then $\overline{0A} + \overline{0B} + \overline{0C} \geq 2(\overline{0X} + \overline{0Y} + \overline{0Z})$ where X is a point on BC and $0X \perp BC$ etc. In 1934 this inequality was proved by L. J. Mordell. As far as is known at present I was lucky enough to find a genuinely new inequality.

Several proofs have been found and many extensions and generalisations. Here I give only a small sample of the relevant literature :

L. J. Mordell, Középisholai Mat. Lapok 11 (1935), 146-148, see also American Math. Monthly, 44 (1937) 252.

L. Fejes-Tóth, Laserungen im der Ebene auf der Kugel und im Raum, Springer Verlag 1953 pages 12 and 28.

§2. In 1932, E. Klein (Mrs. Szekeres) asked : Let $f(n)$ be the smallest integer for which any set of $f(n)$ points $x_1,\dots,x_{f(n)}$ in the plane, no three on a line, always contain a subset $x_{i_1},\dots,x_{i_n}$ which are the vertices of a convex polygon. She proved $f(4) = 5$ and conjectured that $f(n)$ is finite for every n. Szekeres conjectured $f(n) = 2^{n-2} + 1$ and this was proved by Makai and Turán for $n = 5$. Szekeres and I proved

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2}.$$

A few years ago I asked : Is there an n_k so that if $x_1,\dots,x_{n_k}$ are n_k points in the plane, no three on a line then one can always find k of them $x_{i_1},\dots,x_{i_k}$, which form the vertices of a convex polygon which contains none of the x_i in its interior. Trivially $n_4 = 5$ and Ehrenfeucht and Harborth proved that n_5 exists and in fact Harborth proved $n_5 = 10$. It is not yet known if n_6 exists and in fact it is very doubtful if n_k exists for every k.

H. Harborth, Konvexe Fünfecke in Punktmengen, Elemente der Math. 33 (1978), 116-118.

P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935), 463 - 470 and On some extremum problems in geometry, Annales Univ. Sci. Budapest, Sect. Math. 3-4 (1961), 313-320.

See also

P. Erdős, The art of counting, selected writings, M.I.T. Press, 1973.

§3. Let $f(n)$ be the largest integer for which there are n distinct points $x_1,\dots,x_n$ in the plane for which there are $f(n)$ pairs x_i, x_j with $d(x_i,x_j) = 1$, $d(x_i,x_j)$ is the distance between x_i and x_j. I proved that for some positive constants c_1 and c_2

$$n^{1+c_1/\log\log n} < f(n) < c_2 n^{3/2} \qquad (1)$$

and conjectured that the lower bound gives probably the right order of magnitude.

Szemerédi and Józsa proved $\frac{f(n)}{n^{3/2}} \to 0$. I offer 300 U.S. dollars for the proof (or disproof) of my conjecture and would pay already for $f(n) < n^{1+\varepsilon}$ for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$.

Let $g(n)$ be the largest integer for which there always are at least $g(n)$ distinct numbers among the $d(x_i,x_j)$, $1 \le i \le j \le n$. I proved

(2) $$(n-1)^{\frac{1}{2}} - 1 < g(n) < c\,n\,(\log n)^{-\frac{1}{2}}$$

and conjectured that in (2) the upper bound gives the correct order of magnitude. The lower bound in (2) was improved by L. Moser to $c\,n^{2/3}$ which is the best result at present.

I conjectured that if the points $x_1,\dots,x_n$ are the vertices of a convex polygon then $g(n) \ge [\frac{n}{2}]$. This conjecture was proved by Altman. Szemerédi conjectured that the same result holds if we only assume that no three of the x_i are on a line, but he could prove this only with $[\frac{n}{3}]$ instead of $[\frac{n}{2}]$.

P. Erdös, On some problems of elementary and combinatorial geometry, Annali di Math. Ser. IV, V 103 (1975), 99-108. This paper contains extensive references to all the problems discussed here. See also, Some combinatorial problems in geometry, Geometry and differential geometry, Proc. Haifa, Israel 1979, Lecture notes in Math. 792, 46-53, Springer Verlag.

G. Purdy and I plan to write a book on these and related problems.

II.

Here I discuss conjectures in number theory. Not to make the paper too long here I will only discuss conjectures which have been settled recently.

§1. Let $|z_n| = 1$ be an arbitrary sequence of complex numbers. Put

$$A_n = \max_{|z|=1} \left| \prod_{i=1}^{n} (z-z_i) \right|.$$

I conjectured more than 20 years ago that

(1) $$\limsup A_n = \infty.$$

(1) clearly belongs to the subject called irregularities of distribution started by van der Corput and Aardenne-Ehrenfest. Very recently Wagner proved (1) using a method of W. Schmidt.

I expect that for some $c > 0$ and for all $n > n_o(c)$.

$$\sum_{k=1}^{n} A_k > n^{1+c}$$

and that A_n can be bounded for only "very few" values of n.

I stated many problems on diophantine approximations in two of my survey papers.

In the second one I give extensive references to the results obtained on these problems. Thus, not to make this paper too long, I refer here only to these papers. Here I only restate one of the problems which has been settled since then by de Mathan and Pollington (independently) : Let $n_1 < n_2 < \dots$ satisfy $n_{k+1}/n_k > c > 1$. Then there is always an irrational α for which the fractional part of $n_k\alpha$ is not everywhere dense. It turned out that the set of these α's has Hausdorff dimension 1 in every interval.

P. Erdös, Problems and results on diophantine approximations I and II, Compositio Math. 16 (1964), 52-65 and Répartition Modulo 1, Coll. Masseille-Luminy 1974, Lectures notes in math. 475, Edité par G. Rauzy, 89-97.

G. Wagner, Problem of Erdös in Diophantine approximation, Bull. London Math. Soc. 12 (1980), 81-88.

B. de Mathan, Number contravening a condition in density modulo 1, Acta Math.Acad. Sci. Hungar. 36 (1980), 237-241.

A. D. Pollington, On the density of sequence $\{n_k\xi\}$, Illinois J. Math. 23 (1979), 511-515.

§2. Denote by $\pi(x)$ the number of primes not exceeding x. I conjectured that for every $x \geq y$

$$\pi(x + y) \leq \pi(x) + \pi(y). \tag{1}$$

Perhaps it is presumptuous to call (1) my conjecture. Implicitly it is certainly stated in earlier papers of Hardy and Littlewood.

A few years ago Hensley and Richards showed that if the so called prime k-tuple conjecture of Hardy and Littlewood holds then (1) certainly fails. The prime k-tuple conjecture states as follows : Let $\{a_1,a_2,\dots,a_k\}$ be any set of k integers. The necessary and sufficient condition that for infinitely many n all the integers $n+a_j$, $1 \leq j \leq k$ should all be primes is that $\{a_1,\dots,a_k\}$ should not form a complete set of residues for all p. The necessity is of course obvious, the whole difficulty is to prove the sufficiency, and in fact the conjecture is probably unattackable by the methods at our disposal since, in particular, it would imply that there are infinitely many prime twins. Hensley and Richards in fact proved that if the prime k-tuple conjecture holds then there is an absolute constant $c > 0$ so that for every $y > y_o$ there is an x for which

$$\pi(x+y) - \pi(x) - \pi(y) > \frac{cy}{(\log y)^2} . \tag{2}$$

In our paper with Richards we disagreed on a conjecture. I believe that for sufficiently large c and every $x > y$ in (2) the inequality has to be reversed and Richards believes that (2) can hold with arbitrary large c. I expect none of us alive now will know the truth. Straus (correctly in my opinion) observed that (1) was wrongheaded in the first place. The correct form of (1) should have been.

(3) $$\pi(x+y) \le \pi(x) + 2\pi\left(\frac{y}{2}\right).$$

One would expect that $\left(-\frac{y}{2}, \frac{y}{2}\right)$ contains more primes than any other interval of length y since this is not quite true perhaps (3) should be modified to

(3') $$\pi(x+y) \le \pi(x) + \max_t \pi(t, t+y), \quad -y < t \le 0,$$

where $\pi(t, t+y)$ denotes the number of primes in $(t, t+y)$. If (3) or (3') holds then it is immediate that (2) holds as I conjectured. In any case the method of Hensley and Richards can not be used to disprove (3) or (3').

D. Hensley and Ian Richards, Primes in intervals, Acta. Arith. 25 (1974), 375-391.

P. Erdös and Ian Richards, Density functions for prime and relatively prime numbers, Monatshefte für Math. 83 (1977), 99-112.

§3. Rankin proved in 1938 that if $p_1 < p_2 < \dots$ is the sequence of consecutive primes then for infinitely many n

(1) $$p_{n+1} - p_n > c\,L_n \ , \text{ for some constant } c > 0,$$

$$L_n = \frac{\log n \cdot \log\log n \cdot \log\log\log n}{(\log\log\log n)^2} .$$

I offered 10,000 (U.S.) dollars for a proof of

(2) $$\limsup\, (p_{n+1} - p_n) / L_n = \infty .$$

I proved that for infinitely many n

(3) $$\operatorname{Min}\,(p_{n+1} - p_n,\ p_n - p_{n-1}) > c_2 L_n$$

and conjectured that for every k

(4) $$\operatorname*{Min}_{0 \le i < k}\,(p_{n+i+1} - p_{n+i}) > c_k L_n .$$

Very recently Maier proved (4) in a very ingenious way. Nevertheless I am certain that, if we put

$$\operatorname*{Max}_{p_n < x}\ \operatorname*{Min}_{0 \le i < k}\,(p_{n+i+1} - p_{n+i}) = f_k(x),$$

then

(5) $$\lim_{x\to\infty} f_{k+1}(x) / f_k(x) = 0.$$

I can not even prove that the lim inf in (5) is 0. It would be of interest to try to prove (5) for other sequences defined by number theoretic properties, e.g. for the squarefree numbers, but I have not succeeded to get any interesting result so far.

I prove that

$$\text{(6)} \qquad \liminf \frac{p_{n+1} - p_n}{\log n} < 1.$$

No doubt the value of the lim inf is 0. I never could prove

$$\text{(7)} \qquad \liminf \frac{\operatorname{Max}(p_{n+1} - p_n,\ p_n - p_{n-1})}{\log n} < 1,$$

but of course there is no doubt that the value is 0 here too.

R. A. Rankin, The difference between consecutive prime numbers, J. London Math. Soc. 13 (1938), 242-247.

H. Maier, Chains of large gaps between consecutive primes, Advances in Math. 39 (1981), 257-269.

P. Erdös, Problems and results on the differences of consecutive primes, Publicationes Math. Delrecen 1 (1949), 33-37.

P. Erdös, The difference of consecutive primes, Duke Math. J. 6 (1940), 438-441.

§4. Sidon asked me nearly 50 years ago to find a sequence $a_1 < a_2 < \dots$ for which the sums $a_i + a_j$ are all distinct and for which a_k tends to infinity as slowly as possible. Sidon called these sequences B_2 sequences and proved that there is a B_2 sequence with $a_k < ck^4$ and he asked for an improvement. He expected that there should be a B_2 sequence satisfying $a_k < k^{2+\varepsilon}$ for all $k > k_o(\varepsilon)$. I observed that the greedy algorithm gives $a_k < ck^3$ (this was also observed by Chowla and Mian who further conjectured that the greedy algorithm gives $a_k < k^{2+c}$ for some $0 < c < 1$). I conjectured that there is a B_2 sequence satisfying $a_k = o(k^3)$. This modest looking conjecture remained open until very recently Ajtai, Komlos and Szemerédi proved it by a new and ingenious method. Unfortunately their method does not give a B_2 sequence for which $a_k < k^{3-\varepsilon}$ holds for some positive $\varepsilon > 0$.

M. Ajtai, J. Komlós and E. Szemerédi, On finite Sidon sequences, European J. Comb. 2 (1980), 1-11.

For further problems and results on these and related problems see the excellent book of H. Halberstam and K. F. Roth, Sequences, I, Oxford Univ Press 1966 and A. Stökr, Gelöste und ungelöste Fragen über Busen der Natürlichen Zahlenreihe, I and II, J. reine u. angew. Math. 194 (1955), 40-65 and 111-140.

§5. Let $f(n)$ be an additive function for which $f(n+1) - f(n) \to 0$. I proved that then $f(n) = c \log n$. I further conjectured that if $f(n+1) - f(n) < c$ then $f(n) = c_1 \log n + g(n)$ where $g(n)$ is an additive function satisfying $|g(n)| < c_1$. This conjecture was proved and generalised by Wirsing in several beautiful papers.

Let $f(n) = \pm 1$ be a multiplicative function. I conjectured more than forty years ago that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n=1}^{x} f(n).$$

always exists and is 0 if and only if $\sum_{f(p)=-1} \frac{1}{p} = \infty$.

Halász and Wirsing in a series of brilliant papers proved and generalised this conjecture.

G. Halász, On the distribution of additive arithmetic functions, Acta Arith. 27 (1975), 143-152, On the distribution of additive and the mean values of multiplicative functions, Studia Sci. Math. Hung. 6(1971), 221-133 and Acta Math. Acad. Sci. Hungar, 23(1972), 425-432.

E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen II, Acta. Math. Acad. Sci. Hungar, 18(1967), 411-467.

E. Wirsing, A characterisation of log n as an additive arithmetic function, Inst. Naz. di alta Math. Symp. Math. vol. IV., Acad. Press. 1974, 45-57. See also a forthcoming paper of Wirsing given at the Durham conference on number theory in 1979.

I. Ruzsa, On the concentration of additive functions, Acta. Math. Acad. Sci. Hungar. 36(1980), 215-232.

N. G. Tchudakoff, Theory of the characters of number semigroups, J. Indian Math. Soc. 20(1956), 11-15.

§6. $\tau(n)$ denotes the number of divisors of n and $\tau^+(n)$ denotes the number of integers k for which n has a divisor in $(2^k, 2^{k+1})$. One of my very old conjectures states that almost all integers n have two divisors $d_1 < d_2 < 2d_1$. This conjecture is still open. I proved that the density of these integers exists but I could never prove that it is 1. In trying to prove this I conjectured that for almost all n $\tau^+(n)/\tau(n) \to 0$. In a recent paper Tenenbaum and I disproved this conjecture.

In the Journee d'Arithmetique at Luminy 1978 Artérisque 61 p.73 I state : Put $(1 = d_1 < \ldots < d_{\tau(n)} = n$ are the divisors of n)

$$Q(n) = \sum_{i=1}^{\tau(n)-1} d_i / d_{i+1}.$$

Prove that for almost all n $Q(n) \to \infty$. (I once claimed that this would imply my conjecture on $d_i < d_{i+1} < 2d_i$. This was nonsense.) It is trivial that $Q(n) \to \infty$ for almost all n. I hope that I can prove that $Q(n)/\tau(n)$ has a distribution function (this should follow from our work with Tenenbaum). My old conjecture on divisors would only follow if I could prove that for almost all n $Q(n) > \frac{1}{2}\tau(n)$, which is almost certainly false.

P. Erdős and G. Tenenbaum, Sur la structure de la suite des diviseurs d'un entier, Annales de l'Inst. Fourier 31 (1981), 17-37. This paper has many references to the older literature.

In July 1981 at the number theory meeting in Budapest, Tenenbaum and I in fact proved that $Q(n)/\tau(n)$ has a continuous distribution function.

§7. Let $1 \le a_1 < a_2 < \ldots$ be a sequence of density 0. Put $A(x) = \sum_{a_i < x} 1$ and denote by $A_2(x)$ the number of distinct integers of the form $a_i + a_j \le x$ and by $B(x)$ the number of distinct integers of the form $0 < a_i - a_j \le x$. I conjectured that if $A(x) = o(x)$ then

$$\limsup \frac{A_2(x)}{A(x)} \ge 3 \quad \text{and} \quad \limsup \frac{B(x)}{A(x)} = \infty .$$

I observed that the first conjecture if true is best possible. Freiman proved my first conjecture and Ruzsa my second. Ruzsa further proved the following conjecture of mine : There is a sequence of integers $1 \le a_1 < \ldots$ satisfying $\sum_{a_i < x} 1 < \frac{cx}{\log x}$ and every integer is of the form $a_i + 2^k$.

I. Ruzsa, On a problem of P. Erdős, Canada. Math. Bull. 15(1972), 309-310, On difference sequences, Acta Arith. 25(1973/74), 151-157, On the cardinality of A + A and A - A, Coll. Math. Soc. J. Bolyai 18, Combinatorics, 1976 edited by A. Hajnal and V. T. Sós, 933-938.

G. A. Freiman, Foundations of a structural theory of set addition, Vol. 37, translations of math. monographs, Amer. Math. Soc. Providence (R.I.) (1973).

§8. A system of congruences $a_i \pmod{n_i}$, $1 < n_1 < n_2 < \ldots < n_k$ is called covering if every integer satisfies at least one of these congruences. My main unsolved problem here is whether n_1 can be chosen arbitrarily large. This remains unsolved. Selfridge and I asked if all the n_i can be odd. This problem also remains open. I asked : Is it true that for every c there is an n for which $\sigma(n)/n > c$ but there is no covering system whose moduli are the divisors of n ?

This question was recently settled affirmatively in an ingenious paper of Haight. For further results and problems on covering congruences see my booklet with Graham quoted in the introduction.

J. A. Haight, Covering systems of congruences : a negative result, Mathematika 26 (1979), 53-61.

§9. Denote by $N(x,\delta)$ the maximum number of points $p_1,\ldots,p_n$ which can be chosen in a circle of radius x so that the distance between any two of them differs by at least δ from every integer. I conjectured that

$$N(x,\delta) = o(x) \quad \text{and} \quad N(x,\delta) \to \infty .$$

The first conjecture was proved by Sárközy who proved

$$N(x,\delta) < \frac{cx}{\delta^3 \log\log x} .$$

Graham proved the second conjecture, he in fact proved

$$N(x,\tfrac{1}{10}) > \tfrac{1}{10} \log x .$$

Sárközy showed that to every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ so that for every $\delta < \delta(\varepsilon)$

$$N(x,\delta) > x^{\frac{1}{2}-\varepsilon} .$$

Perhaps for every $\varepsilon > 0$ $N(x,\delta) < x^{\frac{1}{2}+\varepsilon}$.

Sárközy, On distances near integers I and II, Studia Sci. Math. Hungar. 11(1976), 37-50 and 105-111.

§10. D. Silverman and I asked the following question : Let $1 \le a_1 < \ldots < a_k \le n$ be a sequence of integers for which none of the sums $a_i + a_j$ is a square. Determine or estimate max k. Trivially max $k \ge \frac{n}{3}$; to see this observe that if $a_k \equiv 1$ (mod 3) then the sums $a_i + a_j$ can never be a square. Can k be substantially larger than n/3 ? Is there an infinite sequence $1 \le a_1 < \ldots$ of integers of density $> \frac{1}{3}$ for which none of the sums $a_i + a_j$ are squares ?

I further asked : Let d be any positive integer, $u_1,\ldots,u_t$, $t > d/3$ are residue classes mod d. Is it then true that for some $1 \le i < j \le t$ $u_i + u_j$ must be a quadratic residue (mod d) ? Massias showed that the answer is negative, he gave 11 residues mod 32 so that none of the sums $u_i + u_j$ are quadratic residues. Lagarias, Odlyzko and Shearer proved that if $d \ne 32k$ then my conjecture is correct. They further proved that in general

$$\max k < (0.48 + o(1))n.$$

I then asked : Let $1 \le n_1 < \ldots$ be an infinite sequence of integers and $1 \le a_1 < a_2 < \ldots$ be an infinite sequence of integers for which $a_i + a_j \ne n_u$ for every choice of the integers i, j, u. When can we assert that the density of the sequence $a_1 < a_2 < \ldots$ is less then $\frac{1}{2}$? This problem is not yet settled.

The result of Lagarias, Odlyzko and Shearer will be published soon.

III

Now I discuss problems in combinatorics and graph theory. To shorten the paper I will only discuss recently solved problems. For the older ones I refer to the survey paper of Kleitman and myself, Katona and Burr.

S. Burr, Generalized Ramsey theory for graphs - A survey, Graphs and Combinatorics, Lecture Notes in Math. 416 Springer Verlag (1974), 52-75.

P. Erdös and Kleitman, Extremal problems among subsets of a set, Proc. second chapel Hill conference on Comb. Math., University of North Carolina, Chapel Hill N. C. 1970, 136-145, see also Discrete Math. 8(1974), 281-294.

G. Katona, Extremal problems for hypergraphs, Combinatorics, Proc. Nato Adv. Institute, Nijenrode, Edited by M. Hall and J. H. van Lint, 1975, D. Reidel Dordrecht, 215-274.

§1. I conjectured that for every $\frac{1}{2} < c < \infty$ there is a function $f(c)$ so that every random graph $G(n;cn)$ contains a path of length at least $f(c)n$ where $f(c) \to 0$ as $c \to \frac{1}{2}$ and $f(c) \to 1$ as $c \to \infty$. All these conjectures were proved by Ajtai, Komlós and Szemerédi.

Rényi and I conjectured that with probability tending to one every $G(n; [(\frac{1}{2}+\varepsilon)n \log n])$ is Hamiltonian. This conjecture was proved by Pósa in a very ingenious way with $c\, n \log n$ instead of $(\frac{1}{2}+\varepsilon)n \log n$. His method was the basis of all future work so far on this subject. The full conjecture was proved soon afterwards by Kurshonov and Komlós-Szemerédi.

M. Ajtai, J. Komlós and E. Szemerédi, The longest path in a random graph , Combinatorica 1 (1981), 1-12.

P. Erdös and A. Rényi, On evolution of random graphs, Publ. Math. Inst. Hung. Acad. Sci. 5(1960), 17-61.

L. Pósa, Hamiltonian cycles in random graphs, Discrete Math. 14(1976), 359-364.

J. Komlós and E. Szemerédi, Limit distribution for the existence of Hamilton cycles in random graphs, to appear in Discrete Mathematics.

§2. Let $m(n)$ be the smallest uniform hypergraph whose edges have size n and which is three chromatic. The problem is to determine or estimate $m(n)$ as well as possible. In the older literature this question was posed in the following way : determine the smallest integer $m(n)$ for which there is a family of $m(n)$ sets of size n which does not have property B. The family of sets $\{A_\alpha\}$ is said to have property B if there is a set S such that $S \cap A_\alpha \neq \emptyset$ and $S \not\supset A_\alpha$ holds for every member of the family. This definition is due to E. Miller and was named property B after F. Burnstein. It is easy to see that this means that the hypergraph with edge set $\{A_\alpha\}$ has chromatic number 2. Hajnal and I have a long paper on this subject. It is trivial that $M(2) = 3$ and easy to check that $m(3) = 7$, $m(4)$ is unknown but $19 \le m(4) \le 23$. I proved

(1) $$c_1 2^n < m(n) < c_2 n^2 2^n$$

and conjectured that

(2) $$\frac{m(n)}{2^n} \to \infty .$$

(2) was recently proved by J. Beck, he proved in fact that $m(n) > cn^{1/3} 2^n$.

An symptotic formula for $m(n)$ is still out of reach. An exact formula for $m(n)$ may not exist (i.e. if it exists it is so complicated that it is not illuminating - like a formula for the n-th prime).

Schütte asked me 20 years ago, is there for every n an $f(n)$ so that there is a tournament (or a complete directed graph) of $f(n)$ players so that every set of n players is beaten by at least one of the players. Schütte observed $f(1) = 3$, $f(2) = 7$ but it seemed difficult to calculate $f(n)$ for $n > 2$ (or even to prove the existence of $f(n)$).

I proved by the probability method that for some $c > 0$

$$(3) \qquad 2^{n+1} - 1 \leq f(n) \leq cn^2 2^n ,$$

and I asked for an improvement of (3). E. and G. Szekeres proved $f(3) = 19$ and $f(n) > cn\,2^n$. At the moment an asymptotic formula for $f(n)$ seems beyond reach.

J. Beck, On 3-chromatic hypergraphs, Discrete Math. 24(1978) 127-137.

I. Spencer, Colouring n-sets red and blue, J. Comb. Theory A 30(1981) 112-113.

P. Erdös, On a problem in graph theory, Math. Gazette, 47(1963), 220-223.

E. Szekeres and G. Szekeres, On a problem of Schütte and Erdös, Math. Gazette, 49(1975), 290-293.

P. Erdös and A. Hajnal, On a property of families of sets, Acta Math. Acad. Sci. Hungar. 12(1961), 87-123.

§3. Many mathematicians investigated recently various aspects and generalisations of Ramsey's theorem. Here I only state a problem of Faudree, Rousseau, Schelp and myself which has recently been settled by J. Beck. Let G_1 and G_2 be two graphs, $\hat{r}(G_1, G_2)$ is the smallest integer for which there is a graph G of $\hat{r}(G_1,G_2)$ edges for which $G \to (G_1,G_2)$. In other words : if one colours the edges of G by two colours in an arbitrary way either colour I contains G_1 or colour II contains G_2. We asked for a determination or estimation of $\hat{r}(P_n,P_n)$ and $\hat{r}(C_n,C_n)$. We expected that (P_n is a path of length n and C_n a cycle of n edges.)

$$\hat{r}(P_n,P_n)/n \to \infty \text{ but } \hat{r}(C_n,C_n)/n^2 \to 0.$$

J. Beck in fact proved :

$$(1) \qquad \hat{r}(P_n,P_n) < C_1 n, \quad \hat{r}(C_n,C_n) < C_2 n.$$

The best possible values of the constants are not yet known. Beck further proved various extensions of (1) for trees.

P. Erdös, R. Faudree, C. Rousseau and R. Schelp, The size of Ramsey number, Periodica Math., Hungar. 9(1978), 145-161.

The results of Beck have not yet been published.

§4. Let $|S| = n$ and $A_i \subset S$, $1 \leq i \leq t_n$ be subsets of S which form a partially balanced block design, i.e. every pair (x,y) of elements of S are contained in one and only one of the A's. If $n = p^2 + p + 1$ where p is a prime or a power of a prime then it is well known that a finite geometry exists i.e., there is a block design with $t_n = n$, $|A_i| = p+1$. V. T. Sós and I conjectured that if $t_n > p^2 + p + 1$ then $t_n \geq p^2 + 2p + 1$. A well known theorem of de Bruijn and myself stated that $1 < t_n < n$ is always impossible and our conjecture seemed interesting because it indicated a further gap in the possible values of t_n.

Doyen, V. T. Sós and I discussed the problem of determining all possible values of t_n. A complete solution of this problem seems out of reach at the moment.

V. T. Sós and I proved our conjecture in some special cases and R. Wilson proved the conjecture in full generality, but several related problems remain open.

N. G. de Bruijn and P. Erdös, On a combinatorial problem, Nederl. Akad. Wetensch, Proc. 51(1948), 1277-1279.

§5. Burr and I conjectured that if k is odd then there is a c_k so that every $G(n; [c_k n])$ has a cycle whose length is congruent to $\ell(\mathrm{mod}\ k)$. $G(n;t)$ is a graph of n vertices and t-edges. Robertson and Burr and I had some partial results but Bollobás proved this conjecture with $c_k = k(k+1)2^k$. The true value of c_k is probably much smaller.

Bollobás and I conjectured that every $G(n; [\frac{n^2}{4}]+1)$ has an edge which is contained in at least $\frac{n}{6}$ triangles, and we observed that, if this is true, it is best possible. For the proof we needed the following further conjecture : Let $m > \frac{n^2}{4}$. Then every $G(n;m)$ contains a triangle (x_1, x_2, x_3) for which

$$(1) \qquad v(x_1) + v(x_2) + v(x_3) \geq \frac{3n}{2},$$

where $v(x)$ is the valency or degree of x.

In fact we formulated a more general conjecture (for $k(r)$ instead of $k(3)$). Edwards proved (1) and he in fact proved our conjecture nearly in its full generality.

B. Bollobás, Cycles modulo k, Bull. London Math. Sco. 9(1977), 97-98.

C. S. Edwards, Complete subgraphs with largest sum of vertex degrees, Coll. Math. Soc. J. Bolyai 18, Combinatorics, Edited by A. Hajnal V. T. Sós, North Holland 1978, 293.

For many further problems and results see the excellent book of B. Bollobás, Extremal graph theory, London Math. Soc. Monographs No.11, Acad. Press 1978.

§6. V. T. Sós and I observed that if $|S| = n$ and $A_i \subset S$ $|A_i| = 3$, $1 \leq i \leq t$ and if we further assume that for $1 \leq i < j \leq t$, $|A_i \cap A_j| \neq 1$ then max $t = n$ equality if and only if $n \equiv 0 \pmod 4$. We further conjectured that if $|S| = n$, $A_i \subset S$, $1 \leq i \leq t_k$, $|A_i| = k$ and $|A_i \cap A_j| \neq 1$ for every $1 \leq i < j \leq k$ then for $n > n_0(k)$

$$\text{(1)} \qquad \operatorname{Max} t_k = \binom{n-2}{k-2}.$$

(1) was proved for $k = 4$ by Katona and by P. Frankl in the general case.

I further conjectured that if $|S| = n$, $A_k \subset S$, $1 \leq i \leq t_n$, and if we further assume that

$$|A_i \cap A_j| \neq r \quad \text{for every } 1 \leq i < j \leq t_n$$

(the size of the A's is not restricted here) and that $\varepsilon n < r < (\frac{1}{2} - \varepsilon)n$, then

$$\text{(2)} \qquad t_n < (2 - \varepsilon)^n.$$

As far as I know (2) is still open though P. Frankl has many interesting results which seem to make (2) more plausible.

P. Frankl, Families of finite sets containing no two intersecting in a singleton, Bull. Australian, Math. Soc. 17(1977), 125-134.

P. Frankl and R. M. Wilson, Intersection theorems with geometric consequences, Combinatorica 1 no. 4(1981).

IV

Here I discuss some of my problems in analysis. Since this paper is already longer than I (and probably the editors) planned I will be very brief.

§1. I first mention some problems in function theory. In a paper written in Hungarian (Some remarks on a paper of Kővári, Mat Lapok 7 (1956), 214-217) the following two problems were raised :

(i) Is there an entire function $f(z)$ for which for every infinite sequence of integers $n_1 < n_2 < \dots$ (n_i) the set $\bigcup_{i=1}^{\infty} S_{n_i}$ is everywhere dense, where S_{n_i} is the set of roots of $f^{(n_i)}(z)$?

(ii) Let $H_1, H_2, \dots,$ be an infinite set of sets. Assume that none of the H_k's has a finite limit point. Does there then exist an entire function $f(z)$ and a sequence $n_1 < n_2 < \dots$ for which $S_{n_k} \supset H_k$ for every k ?

The existence of both of these functions has been proved more than ten years ago[(1)] .

About 40 years ago I asked the following question : Let $f(z)$ be an entire function which is not a polynomial function. Is there a path P tending to infinitly for which for every n

$$|f(z) / z^n| \to \infty ?$$

Boas proved that the answer is affirmative, but as far as I know his proof has not been published.

Huber proved that for every $\varepsilon > 0$ there is a path P_ε for which

$$(1) \qquad \int_{P_\varepsilon} \frac{1}{|f(z)|^\varepsilon} < \infty .$$

I conjectured that there is a path P for which the integral (1) is finite for every $\varepsilon > 0$. As far as I know this conjecture has not yet been settled.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function which is not a polynomial function. Put $M(r,f) = \max_{|z|=r} |f(z)|$, $\mu(r,f) = \max_{n \geq 0} |a_n| r^n$.

Define

$$\alpha(f) = \overline{\lim_{r\to\infty}} \frac{\mu(r,f)}{M(r,f)}, \quad \beta(f) = \underline{\lim_{r\to\infty}} \frac{\mu(r,f)}{M(r,f)} .$$

I conjectured that $\alpha(f) = \beta(f)$ implies that both are 0.

Clunie and Hayman disproved this conjecture. In fact they show that for every A, $0 \leq A \leq \frac{1}{2}$, there is an f(z) for which $\alpha(f) = \beta(f) = A$. They also investigate the class of functions for which my conjecture holds.

A. Huber, On subharmonic functions and differential geometry in the large, Comm. Math. Helvetici, 32(1957/58), 13-72, see p.52.

J. Clunie and W. K. Hayman, The maximum term of a power series, J. d'Analyse Math. 12(1964), 143-186.

§2. Now I discuss some of my problems on polynomials. I stated several problems in my paper, Note on some elementary properties of polynomials, Bull. American Math. Soc. 46(1940), 954-958; also Herzog, Piranian and I stated many more problems of a different kind in our paper, Metric properties of polynomials, J. d'Analyse Math. 6(1958), 125-148.

All the problems stated in my first paper were solved by Saff and Sheil-Small, Kristiansen and Bojanic. Many of the problems stated in the second paper were solved by Pommerenke and Elbert. In order to save space I give only references and mention just two of the problems in the paper with Herzog and Piranian which seem to me to be particularly attractive.

1. Let $f_n(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ be a polynomial of degree n. Is it true that the length of the leminiscate $|f_n(z)| = 1$ is maximal if $f_n(z) = z^n - 1$? This problem is still open and seems to us to be a very nice conjecture.

2. Put $f_n(z) = \prod_{i=1}^{n} (z-z_i)$, $|z_i| \leq 1$, $1 \leq i \leq n$. Denote by $A(f_n)$ the area of the

set $|f_n(z)| \le 1$ and put $\varepsilon_n = \text{Min } A(f_n)$, where the minimum is extended over all polynomials of degree n whose roots are all in the unit circle. We proved that $\varepsilon_n \to 0$ and Pommerenke proved $\varepsilon_n > c/n^2$. What is the true order of magnitude of ε_n ? Perhaps it tends to 0 logarithmically.

E. B. Saff and T. Sheil-Small, Coefficient and integral mean estimates for algebraic and trigonometric polynomials with restricted zeros, J. London Math. Soc. 9(1975), 16-22.

G. K. Kristiansen, Some inequalities for algebraic and trigonometric polynomials, J. London Maths. Soc. 20(1979), 300-314.

The paper of Bojanic has not yet been published, it will appear soon.

§3. Some of my older problems on interpolation have been settled in a series of remarkable papers by Kilgore, de Boor and Pinkus and Bratman - these papers contain also many further results.

Vértesi and I recently proved that for every triangular matrix $\{x_{k,n}\}$

$$(1) \qquad -1 \le x_{n,n} < \dots < x_{1,n} \le 1, \; n = 1,2,\dots$$

there is a continuous function f(x) for which the sequence of Lagrange interpolation polynomials $\mathcal{L}_n(f(x))$ taken at the points (1) diverge almost everywhere. The following curious problems remain open. Put $\omega_n(x) = \prod_{i=1}^{n} (x - x_{i,n})$ and let

$$\ell_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_{k,n})(x - x_{k,n})}$$

be the fundamental functions of the Lagrange interpolation polynomials. Let $\mathcal{B}$ be the set of points x in (-1, +1) for which the sequence $\sum_{k=1}^{n} |\ell_k^{(n)}(x)|$ remains bounded as $n \to \infty$. It is well known that if $x_0 \notin \mathcal{B}$ then there is a continuous f(x) for which the sequence $\mathcal{L}_n(f(x_0))$ diverges, but if $x \in \mathcal{B}$ then $\mathcal{L}_n(f(x)) \to f(x)$.

Is there a triangular matrix with the property that for every continuous f(x) there is an $x_0 \notin \mathcal{B}$ for which nevertheless $\mathcal{L}_n(f(x_0)) \to f(x_0)$? In other words: there is no f(x) for which $\mathcal{L}_n(f(x))$ diverges at all the points where it possibly could diverge.

Is there a matrix (1) for which $\mathcal{B}$ is empty but for every continuous f(x) there is an x_0 for which $\mathcal{L}_n(f(x_0)) \to f(x_0)$? I offer 250 (U.S.) dollars for settling these problems.

A fertile source of interesting problems is a recent paper of P. Turán.

C. de Boor and A. Pinkus, Proof of the conjectures of Bernstein and Erdös concerning the optimal for polynomial interpolation, J. Approximation theory 24(1978), 289-303.

L. Bratman, On the polynomial and rational projections in the complex plane, S.I. A.M.J. of Numerical Analysis (to appear).

P. Erdős and P. Vértesi, On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes, Acta Math. Acad. Sci. Hungar, 36(1980), 71-89.

T. A. Kilgore, A characterisation of the Lagrange interpolatory projection with minimal Chebyshev norm, J. Approximation Theory 24(1978), 273-288.

P. Turán, On some open problems of approximation theory, Journal of Approx Theory 29(1980), 23-89.

V

In this final chapter I discuss miscellaneous problems. First I select two problems on set theory.

§1. Thirty years ago Rado and I started to work on the subject which Rado called partition calculus. One of our first problems was : characterise those α for which if S is a well-ordered set of type ω^{α} and G a graph whose vertices is S then either S has a triangle or if not then G has an independent set of type ω^{α} (i.e. there is a subset $S' \subset S$ of type ω^{α} no two vertices of which are joined).

I hope the reader will permit a very old man to give some personal reminiscences. I seem to remember that Rado and I hoped that it will be possible to characterise the α for which $\omega^{\alpha} \to (\omega^{\alpha}, 3)$ and that perhaps this in fact holds for all α. In November 1954 after the International Congress in Amsterdam, I was in Zurich on the way to Jerusalem and I told E. Specker : I give 20 (U.S.) dollars for a proof or disproof of $\omega^2 \to (\omega^2, 3)$. Specker's proof of this conjecture soon reached me in Jerusalem. Then next summer Specker told me his surprising example $\omega^n \not\to (\omega^n, 3)$ for every $3 \leq n < \omega$. Specker observed that neither his proof nor his counter-example works for $\omega^{\omega} \to (\omega^{\omega}, 3)$ and he called attention to this interesting and surprising difficulty.

I soon offered 250 (U.S.) dollars for a proof or disproof and in 1970 C.C. Chang proved $\omega^{\omega} \to (\omega^{\omega}, 3)$. E. Milner soon somewhat simplified this proof and also showed that $\omega^{\omega} \to (\omega^{\omega}, n)$ holds for every $n < \omega$. Finally Jean Larson independently obtained a considerably simpler proof of $\omega^{\omega} \to (\omega^{\omega}, n)$. She further observed that if $\omega^{\alpha} \to (\omega^{\alpha}, 3)$ then α must be power of ω. The first open problem is :

$$\omega^{\omega^2} \to (\omega^{\omega^2}, 3). \tag{1}$$

I offer 250 (U.S.) dollars for a proof or disproof of (1) and a 1000 (U.S.) dollars for clearing up completely the truth value of $\omega^{\alpha} \to (\omega^{\alpha}, 3)$.

P. Erdős and R. Rado, A partition calculus in set theory, Bull. Amer. Math. Soc.

62(1956), 250-260.

E. Nosal nearly completely settled the truth value of $\omega^n \to (\omega^m, k)$.

E. Specker, Teilmengen von Mengen mit Relationen, Comment. Math. Helv. (1957), 302-314.

C. C. Chang, A partiition theorem for the complete graph on ω^ω, J. Combinatorial Theory, (Ser A) 12(1972), 396-452.

J. A. Larson, A short proof of a partition theorem for the ordinal ω^ω, Ann. Math. Logic 6(1973/74), 129-145.

E. Nosal, Partition relations for denumerable ordinals, J. Comb. Theory, (Ser.B) 27(1979), 190-197.

§2. Let α be an ordinal which has no predecessor (i.e. α is a limit ordinal). Hajnal, Milner and I asked : Let G be a graph whose vertices form a set of type α. Is it true that either G contains an infinite path (it does not have to be monotonic) or it contains an independent set of type α? We proved this for all $\alpha < \omega_1^{\omega+2}$. Our proof breaks down for $\alpha = \omega_1^{\omega+2}$. I offer 250 (U.S.) dollars for settling the problem for $\omega_1^{\omega+2}$ and 500 (U.S.) dollars for the general case.

P. Erdös, A. Hajnal and E. C. Milner, Set mappings and polarised partition relations, Combinatorial theory and its applications, Coll. Math. Soc. J. Bolyai 4, 1969, 327-363, see p.358.

§3. More than 30 years ago (sharpening an unpublished result of Mrs and Mr. Boas) I conjectured : Let f(x) be a real function. Assume that $f(x+h) - f(x)$ is continuous for every $h > 0$ and $-\infty < x < \infty$. Is it true that

$$f(x) = g(x) + h(x)$$

where g(x) is continuous and h(x) is Hamel function, (i.e., $h(x+y) = h(x) + h(y)$)?

I could not settled this conjecture, but I could do the next best thing : I told this to N. G. de Bruijn, I thought that he would be able to settle this problem. I was right : He not only settled it but obtained much more general results.

De Bruijn and I raised the following problem : Assume that $f(x+h) - f(x)$ is measurable for every $h > 0$ and $-\infty < x < \infty$. Is it then true that

$$f(x) = g(x) + h(x) + r(x)$$

where g(x) is continuous, h(x) is Hamel and $r(x+h) - r(x) = 0$ for every h and almost all x ? (The set where $r(x+h) - r(x) \neq 0$ can of course depend on h but it must have measure 0.)

This conjecture which remained open for 30 years was recently proved by Laczkovich.

N. G. de Bruijn, Functions whose differences belong to a given class, Nieuw Archief Voor Wiskunde 23(1951), 194-218.

M. Laczkovich, Functions with measurable differences, Acta Math. Acadm. Sci. Hungar. 35(1980), 217-235.

§4. Finally I discuss my meagre contributions to topology. My most important contribution to topology I owe to World War II : Late in August 1939 Hurewicz asked me to determine the dimension of the rational points in Hilbert space. He was too upset to be able to think about it (A few weeks earlier his parents returned to Poland - against his advice - from a sense of duty. This had a happy ending insofar as anything in life can have a happy ending. They managed to get back to the U. S. in 1940 and died peacefully there, fortunately, before the tragic untimely death of Hurewicz in 1956.)

I soon proved that the dimension of the rational points in Hilbert space is 1.

I asked two problems : Can the rational points in Hilbert space be topologically imbedded in the plane ? Roberts proved that the answer is affirmative.

Is there for every $n > 1$ a space S of dimension n for which S^2 also has dimension n ? I noticed that the rational points in Hilbert space prove this for $n = 1$. This was settled affirmatively about 15 years ago(2).

Nearly 40 years ago I asked the following questions : Is it true that every connected set in Euclidean space contains a connected subset, which is not a point and which is not homeomorphic to it ?

Is it true that every connected set (in a Euclidean space) of dimension greater than one contains more than $c = 2^{\aleph_0}$ connected subsets ?

I was rather pleased with these questions but Eilenberg told me that he does not think that the questions will be very illuminating since a clever and difficult counter-example will be found to both of them. Unfortunately he was right, Mary Ellen Rudin using the continuum hypothesis found the required counter examples.

P. Erdös, The dimension of the rational points in Hilbet space, Annals of Math. 41(1940), 734-736.

J. H. Roberts, The rational points in Hilbert space, Duke Math. J. 22(1956), 489-491.

P. Erdös, Some remarks on connected sets, Bull. Amer. Math. Soc. 50(1944), 442-446.

M. E. Rudin, A connected subset of the plane, Fund. Math. 46(1958), 15-24.

Added after completing the paper. July 1981.

There is here in Eger, Hungary a meeting on combinatorial analysis and this permits some last minute corrections and additions.

I now would like to state two new problems. Let $G(n;e)$ be a graph of n vertices and e edges. We assume that e/n is large. Is it true that there is a function $f(x)$, $f(x)/x \to \infty$ as $x \to \infty$ for which

$$\hat{r}(G(n;e)) > e\, f\left(\frac{e}{n}\right) \ ?$$

Let $G(n)$ be a graph with bounded edge density for all subgraphs. In other words there is an absolute constant c so that if $G(k)$ is any subgraph of G of k vertices then the number of edges of $G(k)$ is less than ck. Burr and I conjectured several years ago that then

$$\hat{r}(G(n)) < f(c)n. \tag{1}$$

In other words the ordinary diagonal Ramsey number of $G(n)$ is less than $C\,n$ where C depends only on c.

Perhaps in fact

$$\hat{r}(G(n)) < f(c)n \tag{2}$$

(2) would clearly imply (1), my first feeling would be to try to find a counter example to (2).

Several years ago I conjectured that if one colours the edges (i,j), $1 \le i < j \le n$, by two colours, then if t is any given number and $n > n_0(t)$ then there is always a monochromatic complete graph having the vertices $1 \le i_1 < \dots < i_k \le n$ for which

$$\sum_{r=1}^{k} \frac{1}{1+\log i_r} > t. \tag{3}$$

The interest of (3) is that it does not follow immediately from Ramsey's theorem. Rödl now proved (3). In fact put

$$\max \sum_{r=1}^{k} \frac{1}{1+\log i_r} = F(G(n)), \tag{4}$$

$$\min_{G} F(G(n)) = F(n)$$

where the maximum in the first case is extended over all monochromatic complete graphs and in the second equation the minimum is extended over all colourations of our complete graph. Rödl proved that

$$c_1 \frac{\log\log\log\log n}{\log\log\log\log\log n} < F(n) < c_2 \log\log\log n.$$

He also showed that (3) fails for colouring with three colours. His paper on this subject will appear soon.

NOTES

(1) K. F. Barth and W. J. Schneider, On a problem of Erdős concerning the zeros of the derivatives of an entire function, Proc. Amer. Math. Soc. 32 (1972) 229-232.

K. F. Barth and W. J. Schneider, On a problem of Bagemihl and Erdős concerning the distribution of zeros of an annular function, J. reine u angew. Math. 234 (1969), 179-183.

K. F. Barth and W. J. Schneider, Entire functions mapping countable dense subsets of the reals onto each other monotonically, J. London Math. Soc. (2) 2 (1970), 620-626.

K. F. Barth and W. J. Schneider, Entire functions mapping arbitrary countable dense sets and their complements onto each other, J. London. Math. Soc. (2) 4 (1972) 482-488.

(2) R. D. Anderson and J. E. Kiesler, An example in dimension theory, Proc. Amer. Math. Soc. 18 (1967), 709-713.