COMBINATORIAL PROBLEMS IN GEOMETRY*

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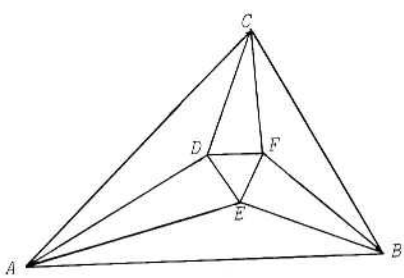
(received 7 September, 1982)

Actually this is my first lecture in New Zealand. Perhaps if I live it won't be the last.

There is a running discussion between Dieudonné and Branko Grünbaum. Dieudonné sort of says that geometry is dead and of course Branko Grünbaum disagrees with him. I think I am on the side of Branko Grünbaum and I hope that I will convince you that at least combinatorial geometry is not dead. Now some sort of classical Euclidean geometry perhaps is sort of semi-dead. I call a subject dead if no new theorems and no new conjectures have been born. Now the last really good theorem in classical Euclidean geometry states as follows - this is a theorem of Morley - it says if you take a triangle (ABC) and you trisect the angles like this:



and you construct the points of intersection $(D, E,$ and $F)$ then this triangle inside is equilateral. Extremely striking and beautiful

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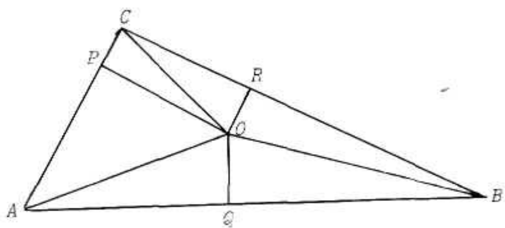
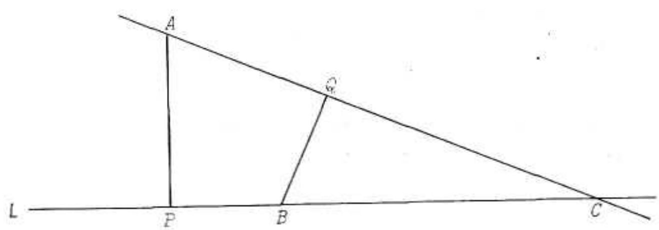
theorem. And I think probably Euclid would have liked this theorem and accepted it. Actually I am not a hundred percent sure because there is trisection and maybe Euclid wouldn't have accepted that. Sometimes I make the joke: maybe soon I will be able to ask him whether he accepts this theorem or not. I won't talk of course about such things.

Now there are some new things in geometric inequalities. One of the first things which I was associated with is the Erdös-Mordell inequality. I conjectured this in 32 - that is in 1932 not 1832 - which says as follows (Mordell proved it two years later): if you have a triangle and you take a point O inside and you drop the perpendiculars and you join it with opposite sides:



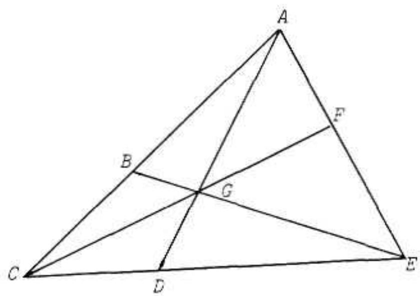
then this (OA) plus this (OB) plus this (OC) is greater or equal to twice the sum of the perpendiculars. Equality occurs only if the triangle is equilateral and the point is in the centre. But I won't talk about geometric inequalities either.

Now let me start about combinatorial geometry; what kind of problems I have in mind? There was a very beautiful book by Hilbert and Cohn-Vossen. It appeared in 1933, the title then was *Anschauliche Geometrie*, the English title is *Geometry and the Imagination*. Well I was reading this in 1933 and suddenly the following conjecture occurred to me. Suppose you have n points in the plane, not all on a line. Then it occurred to me that there must be a line which goes through precisely two of the points. So I repeat here: n points in the plane, not all on a line, then there is a line which goes through exactly two of the points. Or in another way if you have n points in the plane and every line which goes through two of them also goes through a third then the points all lie on a straight line. I thought first I would prove it immediately. I couldn't prove it and a few weeks later a friend of mine in Hungary, Gallai, proved it; and this was the first proof as far as I know. Latter L.M. Kelly found that the conjecture is not new. It was conjectured in 1893 by Sylvester. It appeared in the Educational Times, which seems to have been forgotten, and as far as I know Gallai's proof is the first. Now the simplest proof for this theorem is due to L.M. Kelly and let me present the proof now to you. It comes straight out of the book. I have to explain this. One of my jokes is: God has a transfinite book which all theorems and the best proofs are in and if he is well-intentioned to us he shows us the book for a moment. And I say you don't even have to believe in God but you should believe that the book exists. Now this is the way L.M. Kelly's proof works. Suppose we have n points in the plane and every line through two of them goes through a third and the points are not all on a straight line. We will now come to a contradiction. Take the smallest distance of a point from a line. Maybe there are several but since the number of points is finite there is a smallest distance. Here is the smallest distance (AP):



OK now. This line (L) contains at least three points (because every line through two goes through three) so you can assume there are two points here $(B$ and $C)$. This point (B) may fall in here (*i.e.* may coincide with P). Now join this point (A) to this point (C) and drop this perpendicular (from B to AC). Clearly this distance (BQ) is smaller than this distance (AP). And we arrive at a contradiction. And this is it. It is a very clever and simple proof and you look rather foolish if you try to prove it and didn't think of that.

Now there are many problems which you can raise here. Just to mention one of them, de Bruijn and I stated that if you have n points then the number of lines which go through precisely two of them goes to infinity. We stated this as a conjecture. This was proved by Motzkin in 1951 in the Transactions and then Kelly and Moser proved that there are $3n/7$ lines which go through two; and this is best possible at least for seven because if you draw these lines:

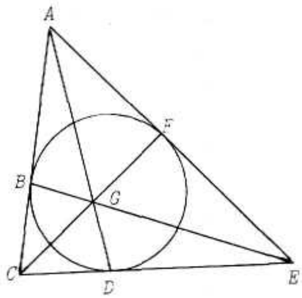


- there are seven points here - and the only lines through two of them are these three lines $(BF, FD, DB,)$. But anyway Motzkin conjectured that for n greater than thirteen there are at least $n/2$ ordinary lines (*i.e.* lines which go through exactly two of the points.) And for even n he showed that this is best possible if true. Very recently a Danish highschool teacher called Hansen proved this conjecture. I haven't seen the proof yet, it is quite new and is really very complicated, it goes on for about 50 pages. Very complicated. But it seems that it was checked by Fenchel carefully and it is correct.

Now even before Gallai proved his theorem I noticed that it has the following nice consequence. Suppose you have n - this goes back you know fifty years now (not five hundred only fifty) - this is the theorem: if you have n points in the plane, not all on a line and you join any two of the points then you get at least n distinct lines. So you have n points in the plane, not all on a line, and you join every two of them, then you get at least n distinct lines. This is best possible if true because if you have n-1 points on a line and one off you get n lines. This is the way the proof goes. At that time Gallai's theorem didn't exist yet but we will use it now. We will use induction. And also we can assume right away that no n-1 of the points lie on a line because in that case you trivially get n lines. So take the line which goes through exactly two of the points and omit this point (one of the two). Then you are left with n-1 points, which don't lie all on a line, therefore they determine n-1 distinct lines by induction. And this is the answer, because this (*i.e.* the line containing exactly two points) is a new line - it has now only one point so it can't coincide with the other n-1 lines. Now I have here the following problem for which I offer $100. Incidentally, inflation has a little bit to do with the price but more importantly if a problem is unsolved for a long time the price slowly rises. The maximum amount of money I had to pay so far is $1000 for a problem in number theory and I have a problem which is $3000. I was asked once what could happen if all your problems were suddenly solved? Well, I certainly would be in a difficulty but then, what would happen to the strongest bank if all the depositors would suddenly ask for their money? It would collapse instantly and it is much more likely that this happens than that all my problems will be solved. So I think I am reasonably safe. So this is the problem. Suppose you have n points, at most n-k on a line, and you join any two of the points. Then I conjecture that the number of distinct lines is greater than ckn, for an absolute constant c which doesn't depend on k and n. But unfortunately I've never been able to prove it. Now for small k Kelly and Moser have a stronger result but for k about $\sqrt{n}$ this fails. The most interesting case would be - for which I would already pay - if you have $2n$ points, at most n on a line, and you join any two of them then you get more than cn^2 distinct lines. This probably has the whole difficulty, this conjecture. Maybe this isn't so difficult but I don't think it is trivial. [15 August, 1982. J. Beck just proved my conjecture, his paper will appear in Combinatoria - he decided to collect the money in Hungarian currency.]

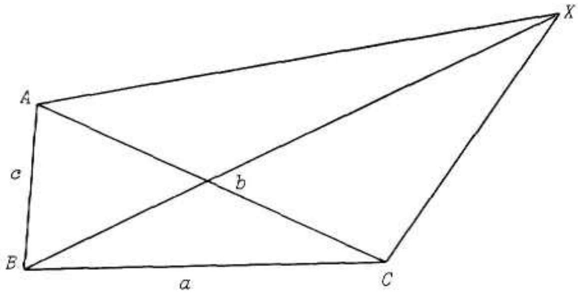
Now about literature on this subject. I have a paper in Annali di Matematica about 1974 and Willy Moser sends out every year from McGill University about unsolved problems in combinatorial geometry. And Branko Grünbaum has several papers on the subject. There's also a very nice book by Hadwiger, Debrunner and Klee which was published in English (originally it was written in German). So there are plenty of ways of finding literature on these problems.

Now this theorem that you get n distinct lines is a special case of a theorem of de Bruijn and myself which is purely combinatorial and which is a generalisation of Fisher's inequality. It states as follows. Suppose you have a set S of n elements and you have a family of subsets A_k contained in S and you know that $2 \leq |A_k| < n$. Now assume that the subsets have the property that every pair of elements is contained in one and only one of the A_k. Then the number of A_ks is at least n. Now the A_ks can be the lines and elements are the points. So once more I repeat: you have a family of subsets $A_1, \ldots, A_m$, of a set of size n, every pair of elements of S is contained in exactly one of the A_ks. Then $n \geq m$. This theorem of mine here is a special case, because here the points are points in the plane and the A_ks are the lines joining two of them. Clearly every pair will be contained in one and only one line. So this purely combinatorial theorem generalises it and it is a genuine generalisation because Gallai's theorem no longer holds here. You know to see that you just have to take the Fano plane. Take all the six lines here:



and add this (the circle) as a seventh line. That is the simplest finite geometry. There are seven elements and seven triples and every pair is contained in one and only one triple. There are many other finite geometries. So let us leave this subject now and talk about something else.

Now this is a metrical problem. This is a joint paper with Anning, an old Canadian mathematician. He was my second collaborator who died. They had once a show in England about 20 years ago, "This is Your Life", you know a radio show, a TV show, where they collected... let us say there was a Resistance leader who helped many people to escape during World War II and they brought together all the people whom he saved. So I thought it would be a good joke if they could bring together all my collaborators. But since there are more than two hundred it would be a little difficult. And it would be especially difficult since so many of them are gone now. So OK, this is the problem which we investigate. Suppose you have an infinite set of points in the plane. And suppose the distance between any two is an integer. Then this is only possible if the points all lie on a straight line. I repeat the theorem: you have an infinite set of points in the plane and suppose every distance between any two of them is an integer. That is only possible if the points are on a straight line. The first proof, a joint proof with Anning, was somewhat complicated but later I found a much simpler proof at the prodding of Kaplansky. Let me present this proof now. So suppose you have an infinite set of points in the plane and all distances are integers. We show that the points must lie on a line. Now if they don't lie on a line then you certainly have a triangle, a non-degenerate triangle. And once you have a non-degenerate triangle then you can have only a finite number of points whose distances from these three points are integers. And in fact the number of points is bounded depending on the triangle. You will see in a moment the proof. Now suppose you take a point X.

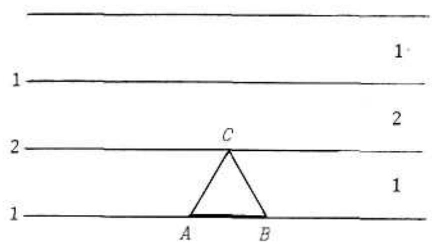


Since XB and XC must be integers you have by the triangle inequality $|XB-XC| \leq a$, so therefore this difference can take on at most $2a+1$ values (it can be from $-a$ to $+a$). That means that the point X must lie on at most $2a+1$ hyperbolas whose foci are these two points (B,C) - if the difference $XB-XC$ is a constant then X is on a hyperbola. The same is true of $XA-XC$, so it also lies on $2b+1$ hyperbolas. Now two such hyperbolas can meet in at most four points, because the foci are different, are non-degenerate, therefore the total number of choices for X is $4(2a+1)(2b+1)$, which is of course finite and only depends on the triangle. I think this proof is also from the book. It is probably the simplest imaginable proof. So once more: we have proved if you have an infinite set of points in the plane and all distances are integers then they must be on a straight line. Now there are several problems which might be asked here. The simplest problem which is unsolved is this : - it is quite easy to find n points on a circle so that all the distances are integers - but can you find n points in general position, no three on a line, no four on a circle, all distances are integers? For example Harborth settled this for $n=5$ but the general case - even for $n=6$ - isn't known. But this is probably difficult because it really belongs to Diophantine approximation. Now the other question is due to Ulam. Can you find a dense set of points in the plane so that all distances are rational? I am almost sure that the answer is no. But this goes back to 1945, so quite a long time. So you have an infinite set of points in the plane which is dense, can it be that all distances are rational? Independently Besicovitch asked the following much weaker question. You have a convex polygon. Can you find arbitrarily close to the vertices - here the convex polygon is an n-gon - can you find arbitrarily close points so that all the distances should be rational? I think that is unsolved even for $n=5$. But here again the real difficulty will not be geometric, this is a problem probably in Diophantine approximation or Diophantine equations and this thing one would perhaps expect to be difficult.

Now let me talk about something completely different. These are problems which we call Euclidean-Ramsey. We have three papers, six of us, Graham, Rothschild, Spencer, Montgomery, Strauss and I. There are several great Montgomerys in mathematics, one is Deane Montgomery, a topologist, and the other is a Montgomery in Michigan. Our Montgomery is comparatively unknown. Anyway it never happened that the six authors were together at the same time yet. So anyway this is the type of problems we considered. Suppose ... let me go back a little bit.

There is a theorem of van der Waerden (maybe I won't be able to tell you anything else but doesn't matter, it is an interesting topic). The theorem of van der Waerden states as follows: if you divide the integers into two classes then at least one of them contains arbitrarily long arithmetic progressions. Here I had to pay $1000 to Szemerédi. Fifty years ago Turán and I conjectured that this problem has really nothing to do with dividing the integers into two classes. If you have a subsequence of integers which has positive density then it already contains an arbitrarily long arithmetic progression. And the proof is very difficult. Szemerédi proved it in 1972 and I offered $1000 for it. And very recently a proof by ergodic theory was obtained by Furstenberg. It is an interesting and growing subject, the application of ergodic theory to number theory. Furstenberg has a book on this subject which recently appeared. Now here is my $3000 problem in this connection. An old conjecture in number theory states that there are arbitrarily long arithmetic progressions among the primes. And I think the only way to approach it is this combinatorial conjecture: if you have an infinite sequence of integers $a_1 < a_2 < \dots$ so that $\sum \frac{1}{a_i} = \infty$, then the sequence contains arbitrarily long arithmetic progressions. This I conjectured more than forty years ago. And since Euler proved that the sum of the reciprocals of the primes diverges, if my conjecture is true it would immediately imply the theorem on primes. So this is a general conjecture : if you have a sequence of integers the sum of whose reciprocals diverges, for every k you can find $a_{i_1}, \dots, a_{i_k}$, which form an arithmetic progression of k terms. Now I offer as I said $3000 for it, and I said I don't think I will ever have to pay this money and I should leave some money for it in case I leave. The second "leave" means of course leave on the journey where you don't need passports and visas. So this has really nothing to do with the topic, it was just an introduction. Now Gallai proved the following generalisation of van der Waerden's theorem. Suppose you have the lattice points in n dimensional space. You give any fixed figure, any finite figure of lattice points, and you divide the lattice points into two classes. Then at least one of the classes contains a figure similar to our figure. This clearly generalises the theorem on arithmetic progressions. Now all work on Euclidean-Ramsey is the following. Suppose you give any finite set S in some Euclidean space. We call it Ramsey if the following situation holds. For every k there is an n_k (n_k depends only on k and on the finite set S) so that if in n_k-dimensional space you divide - you colour by k colours or, if you wish, you divide - the points of n_k-dimensional space into k classes, then at least one class contains a set which is congruent to S. Not similar to S as in Gallai's theorem, but congruent to S. Now the simplest thing which is Ramsey is a square, a unit square. For example in 5-dimensional space you can give fifteen points so that if you colour them by two colours at least one colour contains a unit square. And in general we proved that this (the unit square) is Ramsey: for every k there is an n_k so that you can give a finite set in n_k-dimensional space so that if you colour it by k colours one of them contains a unit square. Now this is all which we know about. We know that every brick, *i.e.* every rectangular parallelepiped, in arbitrary number of dimensions is Ramsey. We also know that every Ramsey set must lie on a sphere. These are the only theorems which we have which are general. The simplest unsolved problem is, for example, is the triangle, an isosceles triangle where one angle is $120°$ and the others are $30°$ and $30°$, is this triangle Ramsey? In other words is it true that if n_k is large enough then in n_k-dimensional space you can find a finite set so that if you split that finite set into k classes then at least one class contains a triangle which is congruent to this triangle? It is surprising how hard this is. We have three papers on this subject, one appeared in the Journal of Combinatorial Theory in 1973 and two papers appeared in a conference report of a meeting in Keszthely, Hungary. This meeting was incidentally held in my memory. By this I mean for my sixtieth birthday.

There are several other problems in this subject which are perhaps of interest. For example we have the following nice conjecture which is really in a way easier to grasp. Is the following theorem true? Suppose you take the plane and you split the plane into two sets S_1 and S_2, so $S_1 \cup S_2$ is the whole plane. Is it true that if you give an arbitrary triangle which is not equilateral then either S_1 or S_2 contains a triangle congruent to this triangle? More generally, if you split the plane into two classes then at least one of the classes contains a triangle which is congruent to any given triangle, with the sole exception of a single equilateral triangle. Now why is that sole exception there? Suppose you make the strip colouring. That is you do this:



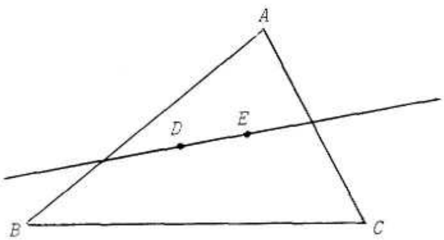
Now if you take this triangle (*ABC*), an equilateral triangle, whose height is this distance (*i.e.* the distance between successive lines), it is easy to see that this triangle does not occur in such a way that the vertices are all coloured 1 or all coloured 2. And our conjecture is that this is the only exception. But we are very far from being able to prove it. Several special cases have been proved. We proved it in our first publication for all triangles whose angles are 120°, 30°, 30°. Many cases have been settled, but the general case, despite its simplicity, is not settled.

Now another problem which we didn't settle is this. (But this has been settled in the meantime.) We had the following conjecture in this paper: suppose you have a set S in the plane and suppose that S has the following property: no two points of S have distance 1. Now is it then true that the complement of S contains four points which form the vertices of a unit square? This was our conjecture. If you have a set S which has the property that no two of its points are of distance 1 , is it then true that the complement of S contains a unit square? Now this was proved by a Hungarian lady called R. Juhász. Actually she proved a stronger theorem. She proved that if - I think it appeared in the Journal of Combinatorial Theory ... (Her husband actually is a man called Csákány. Actually you know some of you may not be familiar with my language. Usually I refer to the husband simply as the slave, the wife I refer to as the boss. But I never lectured in New Zealand before so probably not everyone is aware of this language of mine. So her slave is a well-known Hungarian algebraist called Csákány, Béla Csákány.) She proved the following generalisation: if you have a set S which contains no two points at distance 1 and you give an arbitrary set of four points then there is always in the complement four points which are congruent to them. And she observed also that you can't generalise that for any k. In other words for large k it is not true that for any choice of the points $x_1,\ldots,x_k$ the complement of S contains $y_1,\ldots,y_k$ congruent to $x_1,\ldots,x_k$. Maybe it is still true for 5 but for general k this certainly is not true.

I want to speak about one more problem but before I go there let me mention a nice old question of Steinhaus. You can find it in the problem collection of Willy Moser. Steinhaus asked the following question (this really belongs more to set theory than to elementary geometry). Does there exist a set S in the plane which has the following property: however you put it down in the plane - you take the set S and you translate or rotate it - every set congruent to

S contains exactly one lattice point? So you take the ordinary lattice points, points with integer co-ordinates, does there exist a set S so that every set congruent to S contains exactly one lattice point? I am almost certain that such a set does not exist. I have not been able to prove it. One can reformulate the problem as follows. Take $\sqrt{u^2 + v^2}$, where u and v are integers. Clearly in S you cannot have two points whose distance is $\sqrt{u^2 + v^2}$ because if you would have two points with that distance you could place S in such a way that it should contain two lattice points. So therefore S has the property that no two of its points have this distance. And if this is satisfied then no set congruent to S can contain two lattice points. Now the real problem is can you find such a set which should always contain one lattice point. I am almost certain that it cannot but I have no idea how to prove it. And it is quite possible that this has really a simple solution - I don't think it has but there may be a simple solution. Now there is a classical problem due to Tarski, which is very pretty but this also doesn't belong in elementary geometry. This is the problem of Tarski. You have a square and a circle which has the same area. Can you split this (the square) into a finite number of sets and this (the circle) into a finite number of sets so that A_i is congruent to B_i. In other words can you split the square into a finite number of disjoint sets and you can put it together so that it should be this circle with the same area. It seems to be very difficult but I don't think this is elementary geometry, this is set theory. In 3-dimensional space you can do it, even there you don't have to assume that they have the same volume. That is the famous Banach-Tarski paradox. You can confuse laymen very badly by telling them you should take a dollar bill, you can split it into a finite number of parts and you make two dollar bills out of it. Of course it is obvious cheating because the splitting is not even Lebesgue measurable. So, you know, it is splitting only in the mathematical sense. But anyway it is a very striking result. I am glad to say that Tarski is still alive and kicking, in fact he is getting an honorary degree at Calgary in about two weeks time. He is well over eighty now.

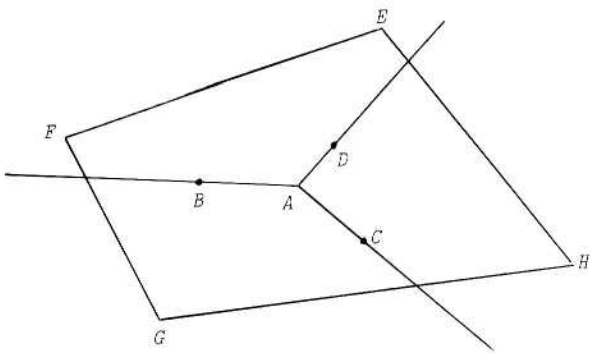
Now, so let me go back to elementary geometry. Again I have to go back about fifty years. In 1931 E. Klein made the following observation. If you have five points in the plane, no three on a line, they always contain the vertices of a convex quadrilateral. Very easy to prove this. Because if you look - (this was once a problem in the Putnam examination much later) - at the least convex polygon, if it is a quadrilateral or a pentagon there is nothing to prove. If it is a triangle (*ABC*) there are two points (*D*,*E*) inside:



you join them and this line (*DE*) doesn't meet one of the sides and here is the convex quadrilateral (*BDEC*). So this has been proved. Then she asked, is it true that to every n there is an $f(n)$ so that if you have $f(n)$ points in the plane, no three on a line, you can always find n of them which form a convex n-gon? So: is it true that there is an $f(n)$ so that if you have $f(n)$ points in the plane, no three on a line, you can always find n of them which form the vertices of a convex n-gon? Now Szekeres and I proved that

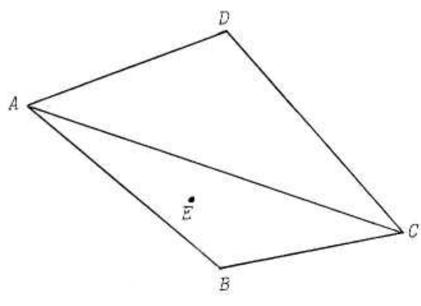
$$2^{n-2} + 1 \leq f(n) \leq \binom{2n}{n}$$

and Szekeres in fact conjectured that the lower bound is the right one. Now I often call this problem the Happy End problem because Esther Klein captured Szekeres and they lived happily ever after in Australia. They are in Sydney now. Actually between 1938 and 58 I of course didn't see them because of the war. I met Szekeres again in University College library in 1958 and since then we have met in many other places, often in Australia. So this problem is still unsolved. Now $f(5) = 9$ - this has been proved by Turán and Makai. I will just tell some of the proof (not the whole proof). If you have nine points in the plane, you take the least convex polygon which contains the points. If this is a pentagon or bigger we have nothing to prove, we have the convex pentagon. It is therefore enough to restrict ourselves to the case where the least convex polygon is either a triangle or a quadrilateral. Now the triangle case is a little messy - this was done by Makai - but Turán has a very ingenious proof if the least convex polygon is a quadrilateral. So then you have a quadrilateral and five points inside. Now if the five points form a convex pentagon there is again nothing to prove, and you have your pentagon. If not then there is a triangle and a point inside:



Now you consider these three lines (AB, AD, AC), they divide the plane into three parts. And there are four points (E,F,G,H) so that at least one part contains two points, and here ($FBADE$) is the convex pentagon. Now it doesn't seem possible to obtain such a proof in general. For $f(6) = 17$ you would need probably some sort of combinatorial reasoning to get that. This we have never been able to do. So this problem is still open.

Now there is the following variant which I noticed when I was once visiting the Szekeres in 1976 in Sydney, the following variant which is of some interest I think. It goes as follows. $n(k)$ is derived as follows, if it exists. It is the smallest integer with the following property. If you have $n(k)$ points in the plane, no three on a line, then you can always find a convex k-gon with the additional restriction that it doesn't contain a point in the interior. You know this goes beyond the theorem of Esther, I not only require that the k points should form a convex k-gon, I also require that this convex k-gon should contain none of the points in its interior. And surprisingly enough this gives a lot of new difficulties. For example it is trivial that $n(4)$ is again 5, that is no problem. Because if you have a convex quadrilateral, if no point is inside we are happy; if from the five points one of them is inside you draw the diagonal (AC):



and you join these (AE, EC) and now this convex quadrilateral $(AECD)$ contains none of the points. And if you have four points and the fifth point is inside then you take this quadrilateral. This is convex again and has no point in the inside. And Harborth proved that $n(5) = 10$. $f(n)$ was 9 in Esther Klein's problem but here $n(5)$ is 10. He dedicated his paper to my memory when I became an archaeological discovery. When you are 65 you become an archaeological discovery. Now, nobody has proved that $n(6)$ exists. What would it mean that it doesn't exist? That you can give, for every t, t points in the plane, no three on a line and such that every convex hexagon contains at least one of the points in its interior. It's perfectly possible that you can do that. Now Harborth suggested to me once that maybe $n(6)$ exists but $n(7)$ doesn't. Now I don't know the answer here.

Let me just tell you one or two problems about distances. The following problem is annoyingly difficult. You have n distinct points in the plane. How many pairs of points can you give whose distance is 1? Let us denote this by $f_2(n)$. Now this problem is surprisingly difficult. I proved in 1946 that

$$f_2(n) < 2n^{3/2}$$

(that also was a Putnam examination problem) and the lattice points in the plane give that

$$f_2(n) > n^{1 + c/\log\log n}$$

This follows from the number of representations of an integer as a sum of two squares. I think the lower bound is the right one. Anyway I couldn't even prove that

$$\frac{f_2(n)}{n^{3/2}} \to 0.$$

I offered \$25 for this and that has been done by Szemerédi about ten years ago, and very recently Beck and Spencer proved that

$$f_2(n) < n^{3/2 - c}$$

So they improved the result of Szemerédi by a good bit. But

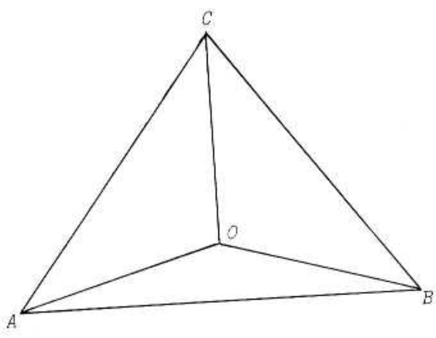
$$f_2(n) < n^{1 + \varepsilon}$$

is nowhere in sight. And I give \$250 for this. Another \$250 for this problem: if you have n points in the plane, how many distinct distances do they determine? I think the answer is

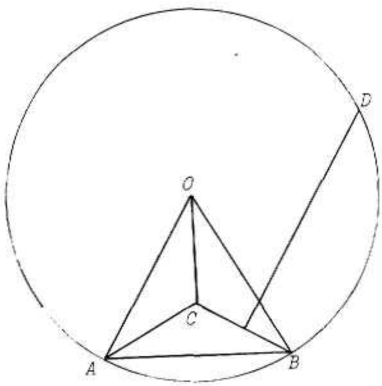
$$\frac{n}{\sqrt{\log n}}$$

- and you get $n/\sqrt{\log n}$ by taking the lattice points in the plane. You see if you take the lattice points in the plane, the distances are $\sqrt{u^2 + v^2}$ and by a theorem of Landau the number of distinct integers of the form $\sqrt{u^2 + v^2}$ is $n/\sqrt{\log n}$. But all I could prove originally was that n points determine $\sqrt{n}$ distinct distances. Now this has been improved by Leo Moser to $n^{2/3}$ and for a long time, thirty years, it didn't change. Very recently Fan Chung, a Chinese lady who works for Bell Labs, improved this to $n^{5/7}$. But the right order I think may be $n/\sqrt{\log n}$. But even the proof of $n^{5/7}$ is very tricky.

Now to end let me ask you a question in elementary geometry. Can you find n points in the plane, no three on a line, no four on a circle, which determine n-1 distinct distances and so that the ith distance occurs i times? By ith distance I mean in any order you wish. For example if you take an isosceles triangle and you take its centre



there are 4 points, 3 distinct distances, and this (OA, OB, OC) occurs 3 times, this (AC, BC) occurs twice and this (AB) occurs one time. Now I mistakenly asserted that I don't believe that for $n > 4$ this is possible. But first Pommerance showed me a construction for $n=5$. This is what he does. You take a circle:



Here are three of the points (O, A, B), an equilateral triangle (the sides are 1, 1, 1), here is the fourth point (C), the centre of the circumscribed circle, and you bisect one of these lines (CB) and here is the fifth point (D). It is easy to see that no three are on a line, no four on a circle; and this distance (OA, AB, OB, OD) occurs four times, this (OC, CA, CB) occurs three times, this (CD, BD) occurs twice and this distance (AD) is unique. And a Hungarian highschool student also showed you can still do it for six points. I am not sure if you can do it for seven points. But I think now my time is up.

[Added 19 August, 1982. Let $h(n)$ be the largest integer so that if n points are given in the plane no three on a line and no four on a circle then they determine at least $h(n)$ distinct distances. Determine or estimate $h(n)$ as well as you can.]

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