Generalizations of a Ramsey-Theoretic Result of Chvátal

Stefan A. Burr
CITY COLLEGE, CUNY, NEW YORK, NY 10031

Paul Erdös
HUNGARIAN ACADEMY OF SCIENCES, HUNGARY

ABSTRACT

Chvátal has shown that if T is a tree on n points then $r(K_k, T) = (k-1)(n-1)+1$, where r is the (generalized) Ramsey number. It is shown that the same result holds when T is replaced by many other graphs. Such a T is called k-good. The results proved all support the conjecture that any large graph that is sufficiently sparse, in the appropriate sense, is k-good.

1. INTRODUCTION

Let F and G be (simple) graphs. Define the *Ramsey number* $r(F,G)$ to be the smallest integer p such that is the edges of the complete graph K_p are colored red and blue, then either the red subgraph contains a copy of F or the blue subgraph contains G. This subject has received much attention lately; see [2], [16] for surveys. One of the most interesting results in the area is the following result of Chvátal.

Theorem 1.1 [11]. If T is a tree on n points, then $r(K_k,T) = (k-1)(n-1)+1$.

Although this theorem is quite easy to prove (in various ways), it seems to occupy a central place in Ramsey theory, since it lends itself to many generalizations and analogs. In this paper we will consider what is perhaps the most direct kind of such generalizations. The following result, which is contained in Theorem 1.1, is a special case of a theory of Chvátal and Harary.

Theorem 1.2 [12]. If G is a connected graph on n points, then $r(K_k,G) \geq (k-1)(n-1)+1$.

Journal of Graph Theory, Vol. 7 (1983) 39–51
 CCC 0364-9024/83/010039-13$02.30

With these theorems in mind, we make the following definition: If G is a connected graph on n points, say that G is *k-good* if

$$r(K_k, G) = (k-1)(n-1)+1.$$

Thus Theorem 1.1 says that for any k, all trees are k-good. In fact, it appears that for fixed k, any large graph that is "sparse" enough is k-good. We will formulate an explicit conjecture of this sort shortly.

If G is a graph define the *edge density* of G as

$$\max_{F \subset G} q(F)/p(F),$$

where the maximum is over all subgraphs of G and where $q(F)$ and $p(F)$ are the number of edges and points of F, respectively. (Any notation not explicitly defined here follows Harary [15]). Of course, the edge density is very close to arboricity, which can be defined as

$$\max_{F \subset G} q(F)/[p(F)-1].$$

We can now state precisely the conjecture mentioned above.

Conjecture. If k and x are fixed, then all sufficiently large connected graphs with edge density no more than x are k-good.

In this article we will begin the systematic study of k-goodness; the problems we examine are suggested by the above conjecture. The conjecture is somewhat related to one made in [6], being that if x is given, then there is a t such that if G has edge density no more then x, the "diagonal" Ramsey number $r(G, G)$ is no more than $tp(G)$. Frequently, the test cases considered and the methods used here are similar to those of [6].

2. KNOWN RESULTS

In this section we state a few results on k-goodness that have been published elsewhere, or soon will be. In addition, we briefly mention some work which is closely related to k-goodness.

Theorem 2.1 [4]. Let k be fixed and let G be a connected graph. If G_1 is a large enough graph homeomorphic to G, then G_1 is k-good.

Actually, the theorem proved in [4] replaces $r(K_k, G)$ by $r(F, G)$, where F is an arbitrary graph. The general theorem involves a notion of "F-goodness," a concept whose definition is slightly technical, so we mention here only the special case represented by Theorem 2.1. In the case that G_1 is a cycle, Theorem 2.1 has already been proved in [1].

By using versions of Theorem 2.1 and Theorem 3.1 from Section 3, and by making certain general upper bounds on $r(K_k, G)$ it is possible to show [8] that all very sparse graphs are k-good. Let $f(k,n)$ be the largest q such that every connected graph on n points and q edges is k-good.

Theorem 2.2 [8]. If $n \geq 4, f(3,n) \geq \frac{1}{15}(17n+1)$. If $\varepsilon > 0$ is fixed, then if n is sufficiently large, $f(3,n) < (\frac{27}{4} + \varepsilon)n \log^2 n$. More generally, if k is fixed, there are constants A and B such that

$$n + An^{2/(k-1)} < f(k,n) < n + Bn^{4/(k+1)} \log^2 n.$$

Another function of interest is $g(k,n)$, the largest q for which there exists any k-good graph with n points and q edges.

Theorem 2.3 [8]. There are constants C and D such that

$$Cn^{3/2} \log^{1/2} n < g(3,n) < Dn^{5/3} \log^{2/3} n.$$

Furthermore, if k is fixed, there are constants C' and D' such that

$$C' n^{k/(k-1)} < g(k,n) < D' n^{(k+2)/k} \log^{\eta} n,$$

where $\eta = 1 - \binom{k}{2}^{-1}$.

In [8], it is also determined that $f(3,3) = g(3,3) = 2$, $f(3,4) = g(3,4) = 5$, $f(3,5) = 7$, $g(3,5) = 8$, and $f(3,6) = 12$. Indeed, in [14], all Ramsey numbers $r(K_3, G)$ are determined for connected G with six points. From this, and from the computations in [10] and [12], the following results can be assembled: If $n = 2,3,4,5$, or 6, a connected graph is 3-good if and only if $G \subset K_2, K_3 - P_2, K_4 - P_2, K_5 - P_3$, or $K_6 - P_4$, respectively. Of course if n is large, can hardly hope for such simple characterizations.

Another result, which extends to 3-goodness only, concerns the following graph $\hat{K}(n)$: Take the complete graph K_n, and for every edge ij, add a new point v_{ij} and edges iv_{ij} and jv_{ij}. Thus $\hat{K}(n)$ has $n + \binom{n}{2}$ points and $3\binom{n}{2}$ edges.

Theorem 2.4 [7]. $\hat{K}(n)$ is 3-good if $n \geq 7$.

It seems very likely that this is actually true for $n \geq 4$. Standard estimates show that $\hat{K}(n)$ cannot be k-good for $k \geq 4$, but probably the subdivision graph of K_n is k-good when n is large.

As has been mentioned, other work related to k-goodness has been done. Some of this involves F-goodness [4], [9], or its multicolor version [3], [13], or hypergraphs [5].

3. TREE-LIKE GRAPHS

In this section we consider what could be thought of as the most basic generalization of Theorem 1.1, since the method is essentially the same as for that theorem, and the results themselves essentially include it. We begin, however, by stating a basic lemma and a theorem pertaining to k-goodness in general.

Lemma 3.1 [4]. For any connected graph G with n points,

$$r(K_k, G) \geq r(K_{k-1}, G) + n - 1.$$

We note that as an immediate consequence we have the following.

Theorem 3.1. If G is k-good, it is $(k-1)$-good.

Recall now that a *free edge* in a graph is an edge with an end point of degree 1. Thus, to *add a free edge* to a graph means to add a point and join it to some point of the graph, by a single edge.

Theorem 3.2. Let G be a connected graph on $n-1$ points, and let G_1 be formed from G by adding a free edge to it. Then

$$r(K_k, G_1) = \max[r(K_k, G), r(K_{k-1}, G_1) + n - 1].$$

Proof. Denote the right-hand side of the above by s. Obviously, $r(K_k, G_1) \geq r(K_k, G)$, and $r(K_k, G_1) \geq r(K_{k-1}, G_1) + n - 1$ by Lemma 3.1. Hence, $r(K_k, G_1) \geq s$. To prove the opposite inequality, consider a two-colored K_s. If we have a red K_k we are done, so we may assume that we have a blue G. Let v be a point of that G such that adding a blue free edge at that point yields a blue G_1. All lines from v to the points not in the G can be assumed to be red, since otherwise we are again done. But $s - (n-1) \geq r(K_{k-1}, G_1)$ points remain outside the G, and either a red K_{k-1} or a blue G_1 gives us a graph we are seeking. This completes the proof. ∎

It seems certain that $r(K_k, G_1)$ almost always equals $\max[r(K_k, G), (k-1)(n-1)+1]$, and this might in fact be true without exception. Any such exception would happen if and only if in the sequence $r(K_2, G) = n - 1$, $r(K_3, G)$, $r(K_4, G), \ldots$, some two terms had a difference of $n-2$, and the preceding two had a greater difference. This might never happen, but at least such a sequence can fail to be convex: $r(K_4, K_3) = 9$, $r(K_5, K_3) = 14$,

$r(K_6,K_3)=18$. In any case we have the following result. It should be pointed out that versions of Theorems 3.2 and 3.3 are also found in [8].

Theorem 3.3. Let G by a connected graph, and form a graph G_n on n points by successive additions of free edges to G. Then if n is large enough, G_n is k-good.

Proof. This is a straightforward double induction on n and k, using Theorem 3.2.

In fact, one can also deduce Theorem 3.3 from the following formula, which is not hard to derive using Theorem 3.2:

$$r(K_k,G_n)=\max[r(K_k,G),\ r(K_{k-1},G)+n-1,\ \ldots,\ r(K_3,G)+(k-3)(n-1),\ (k-1)(n-1)+1].$$

As has already been said, in the usual case all the middle terms of the above should drop out. For instance it is easily checked that if $G=K_3$, then

$$r(K_3,G_n)=\max(6,2n-1),\quad r(K_4,G_n)=\max(9,3n-2),$$
$$r(K_5,G_n)=\max(14,4n-3),\quad r(K_6,G_n)=\max(18,5n-4),$$
$$r(K_7,G_n)=\max(23,6n-5).$$

4. GRAPHS WITH BRIDGES

It is very plausible to guess that if G and H are k-good, so is a graph formed by connecting G and H with a bridge. This is true, and in fact we can prove slightly more.

Theorem 4.1. Let G and H be connected graphs having m and n points, respectively. Suppose G satisfies $r(K_k,G)\le(k-1)(m-1)+3$, and H is k-good. Let F be formed by joining G and H by a bridge. Then F is k-good.

Proof. Since $p(F)=m+n$, we need to prove $r(K_k,F)\le(k-1)(m+n-1)+1$. As always, we can assume $k\ge3$, as well as $m\ge2$, $n\ge1$. Starting with $r(K_k,G)\le(k-1)(m-1)+3$, we may use Lemma 3.1 to deduce other inequalities. Except for the last, all of the following are actually weaker than could be derived.

$$r(K_{k-1},G)\le(k-2)m+1,\quad r(K_{k-2},G)\le(k-3)m+1,\ \ldots,\quad r(K_3,G)\le 2m+1. \tag{1}$$

We also weaken the original inequality to

$$r(K_k, G) \leq (k-1)(m+n-1)+1. \tag{2}$$

We will use these inequalities in place of the original; we could in fact have used them in the statement of the theorem, instead of the simpler but more restrictive one.

Consider now a two-colored complete graph on $(k-1)(m+n-1)+1$ points, and assume it contains neither a red K_k nor a blue F; we will derive a contradiction. Let u and v be the points of G and H, respectively, at which the bridge is to be attached to form F. We will show now that we must have $k-1$ disjoint blue copies $G_1, \ldots, G_{k-1}$ of G, with points $u_1, \ldots, u_{k-1}$, respectively, corresponding to u, and with the edges u_iu_j all being red. We will call this a $(k-1;G)$-configuration, and we will build it up by finding $(l;G)$ configurations (this term having its obvious meaning) successively for $l = 1, 2, \ldots, k-1$.

A $(1;G)$-configuration is just a blue G, so by (2) we have one. Suppose now we have a $(l;G)$-configuration with $1 \leq l \leq k-2$. This has lm points, so that $(k-l-1)m + (k-1)(n-1)+1$ points remain. Since H is k-good, these points span a blue copy of H; designate it H_1, with v_1 being the point corresponding to v. Delete this point; we still have enough to guarantee a blue copy H_2 of H, with its corresponding v_2. We can continue this process until only $(k-1)(n-1)$ points are left. Thus we find $(k-l-1)m+1 = s$ blue copies of H, say $H_1, \ldots, H_s$, with corresponding $v_1, \ldots, v_s$. The v_j are all different, but the H_j generally overlap.

Consider the l points u_i, $i = 1, \ldots, l$. No edge u_iv_j can be blue, since that would yield a blue F; therefore all are red. Now consider the graph spanned by $v_1, \ldots, v_s$. By (1), $r(K_{k-l}, G) \leq (k-l-1)m+1 = s$, unless $l = k-2$, in which case we have $r(K_2, G) = m < s$. Hence, $v_1, \ldots, v_s$ span either a red K_{k-l} or a blue G. But the former is a contradiction, since the red K_{k-l} and $u_1, \ldots, u_l$ would span a red K_k, and the latter yields a $(l+1;G)$-configuration. Thus we see by induction that we must have a $(k-1;G)$-configuration.

This configuration leaves $(k-1)(n-1)+1$ points; these points span a red K_k (a contradiction) or a blue H, say H_1 with its corresponding v_1. But if any edge u_iv_1 is blue, we have a blue F, and if all are red, we have a red K_k. This contradiction completes the proof.

It seems very likely that the following further result is true. Fix k and let $\mathscr{G}$ be a fixed finite set of connected graphs, and let G by a graph formed by attaching members of $\mathscr{G}$ successively by bridges; then if G is large enough, G is k-good. Of course, this is immediate if all members of $\mathscr{G}$ are k-good, in view of Theorem 4.1, but we have not been able to prove the more general result. The result ought to be true, even if our main conjecture is false. A good place to begin on this problem would be with the following graph: Take a large star and attach a copy of K_3 to each endpoint. Is this graph 3-good?

5. POWERS OF GRAPHS

One important test case for our conjecture is that of powers of graphs. Recall that G^t is the graph formed from G by joining with an edge any two points at distance t or less. We will prove that P_n^t is k-good when n is large, and then extend this slightly. The problem is closely related to that of Sec. 5 of [6], and we will begin by stating three lemmas, taken essentially from there. They have been weakened or slightly altered to suit the case at hand.

Lemma 5.1 [6].

$$r(K_k, mK_l) \leq ml + k^{l+1}.$$

Lemma 5.2 [6]. If d and t are given, then there is an l with the following property. If $K_{l,l}$ is two colored, either there is a blue $K_{t,t}$ or there is an $x > l - l/d$ such that at least x points of each part of the $K_{l,l}$ are each joined by red edges to at least x points of the other.

In the above, it is sufficient to take l about as large as td^{t+1}. (The corresponding approximate condition given in [6] is in error.)

Lemma 5.3 [6]. Let G and H be graphs related in the following way: For every point of G there corresponds a set of l points of H. For every edge e of G, the set of edges joining points of the two sets corresponding to the end points of e has the property that at least x points of each set are each joined to at least x points of the other set.

Suppose further that $x > l - l/\Delta(G)$. Then $G \subset H$.

Theorem 5.1. If k and t are fixed, then P_n^t is k-good, when n is large enough.

Proof. We use induction on k. As always, $k = 2$ is trivial. Now assume that the theorem is true for $k - 1$. Consider a two-colored K_s, where $s = (k-1)(n-1) + 1$, and assume, contrary to the theorem, that there is no red K_k and no blue P_n^t; we will derive a contradiction.

Let l be as in Lemma 5.2, with $k - 1$ and $2t$ in place of d and t; also, let l satisfy $l > 2kt$. Then, by Lemma 5.1 there is a c depending only on k and t such that $r(K_k, mK_l) \leq s$, where $m \geq (s - c)/l = [(k-1)(n-1) + 1 - c]/l$.

Therefore, since we have no red K_k by assumption, we have m disjoint blue K_l.

Consider now these m copies of K_l. We form a new two-colored K_m in the following way. Let each point of the K_m correspond to a different one of these K_l, and let each edge of the K_m correspond to the $K_{l,l}$ joining the two K_l in the original graph. If this $K_{l,l}$ contains a blue $K_{2t,2t}$, color the corresponding edge of the K_m blue; otherwise color it red.

We now apply Theorem 1.1 to this K_m. We see that there is a b depending only on k and t so that the K_m either contains a red K_k or a blue P_u, where $u \geq (n-b)/l$. Suppose first that it contains a red K_k. By Lemma 5.2 and the choice of l, we see that for each edge of the K_k we have found, the red edges of the corresponding $K_{l,l}$ satisfy the second alternative of that lemma, with $x > l - l/(k-1)$. But in this case, we can apply Lemma 5.3. Here $G = K_k$, and H is the red subgraph of the $K(l, \ldots, l)$ which corresponds to the red K_k we have found in the K_m. Thus this case leads to a contradiction.

Now turn to the case in which the K_m contains a blue P_u. This P_u yields, in our original K_s, the following blue graph: We have u blue copies of K_l, say $H_1, \ldots, H_u$. Each H_i is joined to H_{i+1} by a blue $K_{2t,2t}$, $i = 1, \ldots, u-1$. By reducing each $K_{2t,2t}$ to a $K_{t,t}$, we can assure that the t points by which H_i is joined to H_{i-1} are disjoint from the t points by which H_i is joined to H_{i+1}. Denote the $l - 2t$ points of H_i which are not points of attachment for H_{i-1} or H_{i+1} by V_i.

This blue graph has $n - a$ points, say, where $a \leq b$, and contains P^t_{n-a}. We wish to work in a more points. Assume that $a \leq u$, as it will be if n is large enough. Suppose a new point v_1 is found which is joined to $2t$ points of V_1 by blue lines. It is not hard to see that this will yield a P^t_{n-a+1}. Similarly, if new points $v_2, \ldots, v_a$ can be found so that v_i is joined to $2t$ points of V_i, we will have the P^t_n we desire.

Suppose we have found a suitable $v_1, \ldots, v_{i-1}$, and are now attempting to find a suitable v_i. There are $s - (n - a + i - 1) \geq (k-2)(n-1) + 1$ points not in the blue graph. We now use, for the only time in the proof, the inductive assumption that $r(K_{k-1}, P^t_n) = (k-2)(n-1) + 1$. From this, we have a red K_{k-1} among these points. Consider now the lines joining this K_{k-1} to V_i. Every point of V_i must have a blue line going to the K_{k-1}, since otherwise we have a red K_k. Hence, at least $l - 2t$ blue lines join V_i to the K_{k-1}. But $l - 2t > (k-1)(2t-1)$, so some point of the K_{k-1} must have at least $2t$ blue lines joining it to V_i. Therefore, if n is large enough the desired $v_1, \ldots, v_a$ all exist, and we have a blue P^t_n, completing the proof. ■

It is interesting to observe that Theorem 5.1 implies Theorem 2.1; this is immediate from the following lemma.

Lemma 5.4. Let G be a fixed graph with p points and q edges. If G_n is a graph with n points which is homeomorphic to G, then $G_n \subseteq P^{p+2q}_n$. This lemma is almost self-evident, but actually carrying out the details is somewhat complicated, so we momentarily defer the proof.

Our purpose in introducing Lemma 5.4 is not really to produce a very complicated proof of Theorem 2.1, but to prove a result which combines Theorems 2.1 and 5.1.

Theorem 5.2. Let k and t be fixed and let G be a connected graph. If G_1 is a large enough graph homeomorphic to G, then G_1^t is k-good.

Proof. Let G have p points and q edges. Then if G_1 has n points, $G_1 \subset P_n^{p+2q}$. But it is clear then that $G_1^t \subset P_n^{t(p+2q)}$. The desired result now follows from Theorem 5.1, with $t(p+2q)$ in place of t. ■

It remains only to give a proof of Lemma 5.4.

Proof of Lemma 5.4. Let $t(p,q)$ be the smallest t such that $G_n \subset P_n^t$ when G_n is any graph with n points which is homeomorphic to a graph with p points and q edges. We must show that $t(p,q) \leq p+2q$ (and therefore that it exists). Actually we will show more, namely that $t(p,q) \leq \max[p, t(p,q-1)+2]$, so that since obviously $t(p,1)=1$, $t(p,2) \leq p$, we have $t(p,q) \leq p+2q-4$, $q \geq 2$. In fact it will be convenient to show as well that the p points of G_n which correspond to the points of G can all be put at one end of the P_n^t.

Suppose now that we have shown this for $q-1$. Without loss of generality, we may assume that G_n be obtained entirely by subdividing edges of G, so that every edge of G corresponds to a path of length at least one in G_n. Let G_m be G_n with one of the paths corresponding to an edge of G removed, leaving m points. Then by hypothesis $G_m \subset P_m^{t(p,q-1)}$, and moreover the points $v_1, \ldots, v_p$ of the path $v_1, \ldots, v_m$ correspond to the points of G. Write t for $t(p,q-1)$.

We now insert some new points. Between v_p and v_{p+1} insert v'_p and v''_p, between v_{p+t} and v_{p+t+1} insert v'_{p+t} and v''_{p+t}, and so on. Suppose that we can continue this process until $n-m$ new points have been inserted. Then any points that were no more than t steps apart are now no more than $t+2$ steps apart. Furthermore, if i and j were the points to be joined by the new path, they can be joined by a path $v_i v'_p v'_{p+t} \cdots v''_{p+t} v''_p v_j$. It is clear that the transition in the middle can be carried out, and that no jump need be greater than $\max(p, t+2)$. If the new points are not enough, simply create enough new points after v_m and drop back to a jump of 2. It is clear that in either case, we have arranged G_n as a subgraph of P_n^{t+2} in the desired way, completing the proof.

It would be of interest to strengthen Lemma 5.4. Very likely $t(p,q)$ is approximately equal to q.

6. WHEELS

An interesting class of graphs with bounded edge density is that of wheels. Here we are only able to show 3-goodness. The notation for wheels is in a confused state, so we offer the following: Denote the n-spoked wheel by $W_{1,n}$. This also allows for multihubbed wheels, by defining $W_{h,n} = \bar{K}_h + C_n$.

Theorem 6.1. Any wheel with at least five spokes is 3-good.

Proof. By a result of [14], $r(K_3, W_{1,5}) = 11$. We prove that $r(K_3, W_{1,n}) = 2n + 1$ when $n \geq 5$ by induction on n. Assume the result true for $n - 1$, and consider a two-colored K_{2n+1}. If there is no red K_3, the induction hypothesis assures us that there is a blue $W_{1,n-1}$. Assume that there is no blue $W_{1,n}$; we will show that this leads to a contradiction.

Let v be the point forming the hub of the blue $W_{1,n-1}$, and let A be the set of points on its rim; denote the points of A by $a_1, \ldots, a_{n-1}$ in order around the rim. Among the remaining points, let B be the set of points connected to v in blue, and C be the set connected to v in red. We make the preliminary observations that $|B| + |C| = n + 1$, that all edges spanned by C are blue, and hence that B is nonempty, since otherwise C would span a ${}_{\text{blue}}K_n$. We now prove a series of facts, leading to a contradiction.

Fact 1. Any point of B is joined to A by at most one blue edge.

To see this, assume to the contrary that for some $b \in B$ and $1 \leq i < j \leq n - 1$, ba_i and ba_j are blue. Clearly, i and j cannot be consecutive (taken modulo $n - 1$), for then we would immediately have a blue $W_{1,n}$. For the same reason, ba_{i+1} and ba_{j+1} must be red. Hence $a_{i+1}a_{j+1}$ must be blue, to avoid a red K_3. But then we have a blue $W_{1,n}$, with hub v and rim $a_1a_2 \cdots a_iba_ja_{j-1} \cdots a_{i+1}a_{j+1}a_{j+2} \cdots a_{n-1}$.

Fact 2. The graph spanned by A contains a ${}_{\text{blue}}K_{n-2}$.

This is immediate from Fact 1, since any two points joined to the same point in red must be joined in blue.

Fact 3. All lines spanned by B are blue.

To see this, simply note that for any $b_1, b_2 \in B$, the lines b_1a_i and b_2a_i must be red for some i, in view of Fact 1.

Fact 4. All lines joining B to C are blue.

To see this assume the contrary, so that bc is red for some $b \in B$, $c \in C$. Without loss of generality, assume ba_i is red for $i = 2, \ldots, n - 1$, so a_ia_j is blue for $2 \leq i < j \leq n - 1$. Then ca_i is blue for $i = 2, \ldots, n - 1$. But this leads to a blue $W_{1,n}$: The hub is a_2, and the rim is $a_1va_3ca_4a_5 \cdots a_{n-1}$.

We can now conclude the proof of the theorem: By Facts 3 and 4, together with the fact that all edges spanned by B are blue, we have that $B \cup C$ spans a blue K_{n+1}, and so we have a blue $W_{1,n}$.

Finally, we observe that $r(K_3, W_{1,3}) = r(K_3, K_4) = 9$ and from [10], $r(K_3, W_{1,4}) = 11$.

7. MULTICOLOR RESULT

Define multicolor Ramsey numbers in the obvious way: If $G_1, \ldots, G_c$ are graphs, $r(G_1, \ldots, G_c)$ is the smallest integer p such that if the edges of K_p are colored in c colors, then, for some i, G_i is contained in color i. Most results on k-goodness can be extended to multiple colors by means of the following result.

Theorem 7.1. Set $k = r(K_{k_1}, \ldots, K_{k_c})$. Then if G has n points and is k-good,

$$r(K_{k_1}, \ldots, K_{k_c}, G) = (k-1)(n-1) + 1.$$

Proof. Denote the left-hand side by r. To see that $r \geq (k-1)(n-1)+1$, color a $K_{(k-1)(n-1)}$ in the following way. Color $k-1$ disjoint copies of K_{n-1} in color $c+1$. The remaining graph has chromatic number $k-1$. But by a result of Lin [17], this graph can be colored in c colors without a K_{k_i} in color i for any i, so we are done. To see that $r \leq (k-1)(n-1)+1$, identify the first c colors momentarily. Since G is k-good, we have either a G in color $c+1$ or a K_k in the first c colors; but by the definition of k, we would then have a K_{k_i} in color i for some i. This complete the proof. ■

Of course, the following is an immediate corollary of this and Theorem 1.1.

Theorem 7.2. If $k = r(K_{k_1}, \ldots, K_{k_c})$ and T is a tree on n points, then

$$r(K_{k_1}, \ldots, K_{k_c}, T) = (k-1)(n-1) + 1.$$

Naturally, these results are fully effective only in the eight cases in which $r(K_{k_1}, \ldots, K_{k_c})$ is known.

8. OPEN PROBLEMS

The biggest open problem here is that of settling the conjecture of Sec. 1. The authors offer a total of \$25.00 (U.S.) for settling it. Our money is probably safe, since it seems very difficult. Various other problems are mentioned in other sections, and we will not call attention to them again; but we mention some other problems suggested by results in those sections.

It would be of considerable interest to narrow the gaps of Theorems 2.2 and 2.3. In particular, does $f(3,n)/n \to \infty$ as $n \to \infty$? In Sec. 4, what can be said if F is formed by identifying a point of G and a point of H? Several problems are also suggested by Section 6. Obviously, it would be desirable to show that for each k, all large wheels are k-good. It would also be interesting to consider multihubbed wheels. Even more interesting might be to combine Theorems 2.1 and 6.1 by replacing a wheel with $G_1 + K_1$, where G_1 is as in Theorem 2.1. One might even be able to combine Theorems 5.2 and 6.1 in a similar way, but this seems very difficult.

Another problem worthy of study is that of cubes. If k is fixed, are all large cubes k-good? Since the cubes have unbounded edge density, this would go beyond the conjecture of Sec. 1. Of course, Theorem 2.3 already shows that that conjecture does not tell the whole story. A possibly related question is the following. Let $\{G_n\}$ be a sequence of k-good graphs with bounded edge density and order going to infinity. Is it true that $G_n \times K_2$ is k-good when n is large enough? One could also ask about other products and other operations on graphs.

Another area of importance is that of general inequalities for $r(K_k,G)$, especially upper bounds. For instance, in [8] it is shown that if $k \geq 3$ and G has p points and q edges, then

$$r(K_k,G) \leq (p+2q)^{(k-1)/2}.$$

Other similar results are to be found in [6]. Such results are useful, for instance, in determining how large n must be for Theorem 3.3 to apply. The same is true for Theorem 2.1. Even less is known about bounds for $r(F,G)$ in general; such bounds have direct application to F-goodness.

We have considered here only connected G. However, there is a natural way to extend the definition of k-goodness to disconnected graphs. Stahl [18] has evaluated $r(K_k, F)$, where F is any forest. The result depends only on k and the orders of the components of F, and therefore provides an appropriate definition of k-goodness for completely arbitrary graphs. This could be a very interesting direction to pursue.

References

[1] J. A. Bondy and P. Erdős, Ramsey numbers for cycles in graphs. *J. Combinatorial Theory Ser. B* 14 (1973) 46–54.

[2] S. A. Burr, Generalized Ramsey theory for graphs—A survey. *Graphs and Combinatorics*. Lecture Notes in Mathematics 406, Springer, Berlin (1974), pp. 52–75.

[3] S. A. Burr, Multicolor Ramsey number involving graphs with long suspended paths. *Discrete Math.* 40 (1982) 11–20.

[4] S. A. Burr, Ramsey numbers involving graphs with long suspended paths, *J. London Math. Soc.* (2) 24 (1981) 405–413.

[5] S. A. Burr and R. A. Duke, Ramsey numbers for certain k-graphs. *J. Aust. Math. Soc. Ser. A* 30 (1981) 284–296.

[6] S. A. Burr and P. Erdös, On the magnitude of generalized Ramsey numbers for graphs. *Institute and Finite Sets, Colloq. Math. Soc. Jànos Bolyai* 10. North-Holland, Amsterdam (1975) 215–240.
215–240.

[7] S. A. Burr and P. Erdös, Generalized Ramsey numbers involving subdivision graphs and related problems in graph theory. *Ann. Discrete Math.* 9 (1980) 37–42.

[8] S. A. Burr, P. Erdös, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, An extremal problem in generalized Ramsey theory. *Ars Combin.* 10 (1980) 193–203.

[9] S. A. Burr and J. W. Grossman, Ramsey numbers of graphs with long tails. *Discrete Math.* 41 (1982) 223–227.

[10] M. Clancy, Some small Ramsey numbers. *J. Graph Theory* 1 (1977) 89–91.

[11] V. Chvátal, Tree-complete graph Ramsey numbers. *J. Graph Theory* 1 (1977) 93.

[12] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, III. Small off-diagonal numbers. *Pac. J. Math.* 41 (1972) 335–345.

[13] P. Erdös, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, Generalized Ramsey theory for multiple colors. *J. Combinational. Theory Ser. B* 20 (1976) 250–264.

[14] R. J. Faudree, C. C. Rousseau, and R. H. Schelp, All triangle-graph Ramsey numbers for connected graphs of order six. *J. Graph Theory* 4 (1980) 293–300.

[15] F. Harary, *Graph Theory*. Addison-Wesley, Reading MA (1969).

[16] F. Harary, The foremost open problems in generalized Ramsey theory. *Proceedings, Fifth British Combinatorial Conference, Aberdeen, 1975*, Utilitas, Winnipeg (1976), pp. 269–282.

[17] S. Lin, On Ramsey numbers and K_r-coloring of graphs. *J. Combinatorial Theory Ser. B* 12 (1972) 82–92.

[18] S. Stahl, On the Ramsey number $r(F,K_m)$ where F is a forest. *Canad. J. Math* 27 (1975) 585–589.