ON ALMOST DIVISIBILITY PROPERTIES OF SEQUENCES OF INTEGERS. I

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1. Throughout this paper we put $e^{2\pi i\alpha}=e(\alpha)$. We write $\{\alpha\}=\alpha-[\alpha]$ and $\|\alpha\|=$ $=\min(\{\alpha\}, 1-\{\alpha\})$ (i.e., $\|\alpha\|$ denotes the distance from α to the nearest integer). $c, c_1, c_2, \dots$ denote positive absolute constants.

We may say that the positive real number b is *almost divisible* by the positive real number a if $\left\|\frac{b}{a}\right\|$ is "small". More exactly, we may say that if $\varepsilon>0$, $\left\|\frac{b}{a}\right\|<\varepsilon$ then b is *ε-divisible* by a and a is an *ε-divisor* of b; in this case, we write $a|_\varepsilon b$.

The aim of this series is to study the ε-divisibility properties of sequences of integers. In particular, in this paper we study *the ε-divisibility by consecutive integers*.

2. In Section 3, we show that if t is not much greater than n, then there exists an integer j such that

$$1 \leqq j \leqq n \tag{1}$$

and $(n+j)|_\varepsilon t$. In fact, Theorem 2 in Section 3 contains this assertion in a sharper form, namely the interval (1) is replaced there by a smaller interval of the form

$$1 \leqq j \leqq P(n, t) \tag{2}$$

(where $P(n, t)$ is much less than n).

Theorem 2 will be derived from Theorem 1 below; this section is devoted to the proof of Theorem 1.

THEOREM 1. *There exists a positive absolute constant c_1 such that the following assertion holds:*

Let $\varepsilon>0$, n a positive integer satisfying $n>n_0(\varepsilon)$, t a real number such that

$$n^2 \leqq t < \exp\left(\frac{(\log n)^{5/4}}{\log\log n}\right). \tag{3}$$

Let us write

$$k=\begin{cases}\left[2\dfrac{\log t}{\log n}\right]-3 & \text{if } 2 \leqq \dfrac{\log t}{\log n} < c_1\\[2ex] \left[\dfrac{\log t}{\log n}+\dfrac{1}{2}\right] & \text{if } \dfrac{\log t}{\log n} \geqq c_1,\end{cases} \tag{4}$$

$$P=\begin{cases}\left[n^{1-1/2^{k+2}}\right] & \text{if } 2 \leqq \dfrac{\log t}{\log n} < c_1\\[2ex] \left[\left(\dfrac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] & \text{if } \dfrac{\log t}{\log n} \geqq c_1\end{cases} \tag{5}$$

(*note that for* $\frac{\log t}{\log n} \geqq c_1$, *i.e.,* $t \geqq n^{c_1}$, *we have* $\frac{1}{2} n^{2/(k+2)} < P \leqq n^{3/(k+2)}$ *by (32) and (33)) and*

$$N(\alpha, \beta) = \sum_{\substack{1 \leqq j \leqq P \\ \alpha \leqq \left\{\frac{t}{n+j}\right\} < \beta}} 1 \quad (for\ \ 0 \leqq \alpha < \beta \leqq 1). \tag{6}$$

Then we have

$$|N(\alpha, \beta) - (\beta - \alpha)P| < \varepsilon P \quad for \quad 0 \leqq \alpha < \beta \leqq 1. \tag{7}$$

PROOF OF THEOREM 1. The proof will be based mostly on Vinogradov's ideas; see [3] and [4]. We need three lemmas.

LEMMA 1. *Let* α, β, Δ *be real numbers satisfying*

$$0 < \Delta < 1/2, \quad \Delta \leqq \beta - \alpha \leqq 1 - \Delta. \tag{8}$$

Then there exists a periodic function $\psi(x)$, *with period* 1, *satisfying*

(i) $\psi(x) = 1$ *in the interval* $\alpha + \frac{1}{2}\Delta \leqq x \leqq \beta - \frac{1}{2}\Delta$,

(ii) $\psi(x) = 0$ *in the interval* $\beta + \frac{1}{2}\Delta \leqq x \leqq 1 + \alpha - \frac{1}{2}\Delta$,

(iii) $0 \leqq \psi(x) \leqq 1$ *in the remainder of the interval* $\alpha - \frac{1}{2}\Delta \leqq x \leqq 1 + \alpha - \frac{1}{2}\Delta$,

(iv) $\psi(x)$ *has an expansion in Fourier series of the form*

$$\psi(x) = (\beta - \alpha) + \sum_{m=1}^{+\infty} (a_m \cos 2\pi m x + b_m \sin 2\pi m x)$$

where

$$|a_m| \leqq \frac{2}{\pi m}, \quad |b_m| \leqq \frac{2}{\pi m},$$

$$|a_m| \leqq 2(\beta - \alpha), \quad |b_m| \leqq 2(\beta - \alpha),$$

$$|a_m| < \frac{2}{\pi^2 m^2 \Delta}, \quad |b_m| < \frac{2}{\pi^2 m^2 \Delta}.$$

This lemma is identical with the special case $r = 1$ of Lemma 12 in [4], p. 32.

LEMMA 2. *Let* r, M, M' *be positive integers,* u, w *real numbers such that*

$$u \geqq 2^{r+3}, \tag{9}$$

$$0 \leqq w \leqq 1, \tag{10}$$

$$M^{\frac{r+3}{2}} \leqq u \leqq M^{r+2} \tag{11}$$

and

$$M \leqq M' \leqq 2M. \tag{12}$$

Then we have

$$\left|\sum_{m=M}^{M'} e\left(\frac{u}{m+w}\right)\right| < c_2 M^{1-1/2^{r-1}-1/2^{r-1}(r+1)} u^{1/2^{r-1}(r+1)} \log u \tag{13}$$

where c_2 is an absolute constant (independent of r, M, M', u, w).

This lemma can be proved by using Weyl's method and it is identical with Theorem 1 in [5], p. 22.

LEMMA 3. *There exists an absolute constant c_3 such that if k, P are positive integers, Q is an integer, $\alpha, \alpha_k, \ldots, \alpha_0$ are real numbers,*

$$k \geqq c_3 \tag{14}$$

and

$$0 < 2(k+1)P|\alpha| \leqq 1, \tag{15}$$

then writing

$$f(x) = \alpha x^{k+1} + \alpha_k x^k + \ldots + \alpha_1 x + \alpha_0,$$

we have

$$\left|\sum_{n=Q+1}^{Q+P} e\big(f(n)\big)\right| \leqq 2e^{15k\log^2 k} P^{1-1/6k^2\log k} \log P + 2|\alpha|^{-1/k}. \tag{16}$$

This lemma can be derived from an estimate of Hua (see [1]), and it is identical with Theorem 4.2 in [2], p. 286.

Now we are going to show that the assertion of Theorem 1 holds with

$$c_1 = \max\left(c_3 + \frac{1}{2},\ \ 20\right)$$

(where c_3 is defined in Lemma 3).

In order to prove (7), we may assume that $\varepsilon < 1$ and let η, ϱ be arbitrary real numbers satisfying $0 \leqq \eta < \varrho \leqq 1$ and

$$\frac{\varepsilon}{4} \leqq \varrho - \eta \leqq 1 - \frac{\varepsilon}{4}. \tag{17}$$

Then writing

$$\alpha = \eta - \frac{\varepsilon}{16}, \quad \beta = \varrho + \frac{\varepsilon}{16}, \quad \Delta = \frac{\varepsilon}{8}, \tag{18}$$

we have $0 < \Delta = \frac{\varepsilon}{8} < \frac{1}{2}$ and

$$\Delta < \frac{\varepsilon}{4} \leqq \varrho - \eta < \beta - \alpha = \left(\varrho + \frac{\varepsilon}{16}\right) - \left(\eta - \frac{\varepsilon}{16}\right) = (\varrho - \eta) + \frac{\varepsilon}{8} \leqq$$

$$\leqq \left(1 - \frac{\varepsilon}{4}\right) + \frac{\varepsilon}{8} = 1 - \frac{\varepsilon}{8} = 1 - \Delta,$$

so that (8) holds and thus Lemma 1 can be applied with the numbers α, β, Δ defined by (18). We obtain that there exists a periodic function $F(x)$ with period 1, satisfying

$$F(x) = 1 \text{ for } \eta \leqq x \leqq \varrho, \tag{19}$$

$$F(x) = 0 \text{ for } \varrho+\frac{\varepsilon}{8} \leqq x \leqq 1+\eta-\frac{\varepsilon}{8}, \tag{20}$$

$$0 \leqq F(x) \leqq 1 \quad \text{for all } x, \tag{21}$$

and such that it has an expansion in Fourier series of the form

$$F(x) = \left(\varrho-\eta+\frac{\varepsilon}{8}\right)+\sum_{m=1}^{+\infty}(a_m \cos 2\pi mx+b_m \sin 2\pi mx) = \tag{22}$$

$$= \left(\varrho-\eta+\frac{\varepsilon}{8}\right)+\sum_{m=1}^{+\infty}\operatorname{Re}(a_m-ib_m)e(mx) = \left(\varrho-\eta+\frac{\varepsilon}{8}\right)+\sum_{m=1}^{+\infty}\operatorname{Re} d_m e(mx)$$

where

$$|d_m|=|a_m-ib_m|=(|a_m|^2+|b_m|^2)^{1/2} \leqq \frac{2\sqrt{2}}{\pi m} < \frac{1}{m}, \tag{23}$$

$$|d_m| = |a_m-ib_m| = (|a_m|^2+|b_m|^2)^{1/2} \leqq 2\sqrt{2}(\beta-\alpha) \tag{24}$$

and

$$|d_m| = |a_m-ib_m| = (|a_m|^2+|b_m|^2)^{1/2} < \frac{2\sqrt{2}}{\pi^2 m^2 \Delta} = \frac{16\sqrt{2}}{\pi^2}\frac{1}{\varepsilon m^2} < \frac{3}{\varepsilon m^2}. \tag{25}$$

Let

$$m_0=\left[\frac{48}{\varepsilon^2}\right]+1.$$

Then by (19), (21), (22), (23) and (25), we have

$$N(\varrho,\eta) = \sum_{\substack{1\leqq j\leqq P \\ \eta\leqq\left\{\frac{t}{n+j}\right\}<\varrho}} 1 \leqq \sum_{1\leqq j\leqq P} F\left(\left\{\frac{t}{n+j}\right\}\right) = \tag{26}$$

$$= \sum_{j=1}^{P} F\left(\frac{t}{n+j}\right) = \sum_{j=1}^{P}\left(\varrho-\eta+\frac{\varepsilon}{8}+\sum_{m=1}^{+\infty}\operatorname{Re} d_m e\left(m\frac{t}{n+j}\right)\right) =$$

$$= \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\operatorname{Re}\left(d_m\sum_{j=1}^{P}e\left(m\frac{t}{n+j}\right)\right)+\sum_{j=1}^{P}\sum_{m=m_0+1}^{+\infty}\operatorname{Re} d_m e\left(m\frac{t}{n+j}\right) \leqq$$

$$\leqq \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}|d_m|\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\sum_{j=1}^{P}\sum_{m=m_0+1}^{+\infty}|d_m| \leqq$$

$$\leqq \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+P\sum_{m=m_0+1}^{+\infty}\frac{3}{\varepsilon m^2} <$$

$$< \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\sum_{m=m_0+1}^{+\infty}\frac{1}{(m-1)m} =$$

$$= \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\sum_{m=m_0+1}^{+\infty}\left(\frac{1}{m-1}-\frac{1}{m}\right)=$$

$$= \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\frac{1}{m_0}<$$

$$< \left(\varrho-\eta+\frac{\varepsilon}{8}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|+\frac{3}{\varepsilon}P\frac{1}{48/\varepsilon^2}=$$

$$= \left(\varrho-\eta+\frac{3\varepsilon}{16}\right)P+\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|.$$

Now we are going to show that

$$\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|<\frac{\varepsilon}{16}P. \tag{27}$$

We have to distinguish two cases.

Case 1. Assume first that $\frac{\log t}{\log n}<c_1$ (i.e., $t<n^{c_1}$). In this case, we are going to apply Lemma 2 with $k, n, n+P-1, mt$ and 1 in place of $r, M\ M', u$ and w respectively. In fact, for large n, (9), (10) and (12) hold trivially. (Note that $k>0$ follows from (3).) Furthermore, we have

$$u=mt\geqq t=n^{\frac{\log t}{\log n}}=n^{\frac{1}{2}\left(\left(2\frac{\log t}{\log n}-3\right)+3\right)}\geqq n^{\frac{1}{2}(k+3)}$$

and for large n,

$$u=mt\leqq m_0t<\frac{49}{\varepsilon^2}t=\frac{49}{\varepsilon^2}n^{\frac{\log t}{\log n}}=\frac{49}{\varepsilon^2}n^{\frac{1}{2}\left(2\frac{\log t}{\log n}-3\right)+\frac{3}{2}}\leqq\frac{49}{\varepsilon^2}n^{k+\frac{3}{2}}<n^{k+2}$$

so that also (11) holds. Thus we may apply Lemma 2. We obtain for lage n that

$$\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P}e\left(\frac{mt}{n+j}\right)\right|<\sum_{m=1}^{m_0}\frac{1}{m}c_2n^{1-1/2^{k-1}-1/2^{k-1}(k+1)}(mt)^{1/2^{k-1}(k+1)}\log mt<$$

$$<\sum_{m=1}^{m_0}c_2\frac{1}{m}n^{1-1/2^{k-1}}\left(\frac{t}{n}\right)^{1/2^{k-1}(k+1)}m\log mt<$$

$$<c_2m_0n^{1-1/2^{k-1}}n^{\left(\frac{\log t}{\log n}-1\right)/2^{k-1}(k+1)}\log m_0t<$$

$$<\frac{c_4}{\varepsilon^2}n^{1-1/2^{k-1}+\left(\frac{1}{2}\left(2\frac{\log t}{\log n}-3\right)+\frac{1}{2}\right)/2^{k-1}(k+1)}\log\frac{49}{\varepsilon^2}n^{c_1}<$$

$$<\frac{c_5}{\varepsilon^2}n^{1-1/2^{k-1}+\left(\frac{1}{2}(k+1)+\frac{1}{2}\right)/2^{k-1}(k+1)}\log n=$$

$$=\frac{c_5}{\varepsilon^2}n^{1-1/2^{k-1}+1/2^k+1/2^k(k+1)}\log n\leqq\frac{c_5}{\varepsilon^2}n^{1-1/2^{k-1}+1/2^k+1/2^{k+1}}\log n=$$

$$=\frac{c_5}{\varepsilon^2}n^{1-1/2^{k+1}}\log n=\frac{c_5}{\varepsilon^2}n^{1-1/2^{k+2}}n^{-1/2^{k+2}}\log n<\frac{\varepsilon}{16}\left[n^{1-1/2^{k+2}}\right]=\frac{\varepsilon}{16}P$$

which completes the proof of (27) in this case.

Case 2. Assume that

$$\frac{\log t}{\log n} \geqq c_1 = \max\left(c_3 + \frac{1}{2},\ 20\right). \tag{28}$$

Let us write

$$f_m(x) = \sum_{l=0}^{k+1} (-1)^l \frac{mt}{n^{l+1}} x^l.$$

Then by the well-known inequality

$$|1 - e(\alpha)| \leqq 2\pi|\alpha|,$$

we have

$$\sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(m \frac{t}{n+j}\right) \right| = \sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(\frac{mt}{n} \frac{1}{1+\frac{j}{n}}\right) \right| = \tag{29}$$

$$= \sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(\sum_{l=0}^{+\infty} (-1)^l \frac{mt}{n^{l+1}} j^l\right) \right| =$$

$$= \sum_{m=1}^{m_0} \frac{1}{m} \left| \sum_{j=1}^{P} e\left(f_m(j) + \sum_{l=k+2}^{+\infty} (-1)^l \frac{mt}{n^{l+1}} j^l\right) \right| \leqq$$

$$\leqq \sum_{m=1}^{m_0} \frac{1}{m} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \sum_{j=1}^{P} \left| e\left(f_m(j) + \sum_{l=k+2}^{+\infty} (-1)^l \frac{mt}{n^{l+1}} j^l\right) - e\big(f_m(j)\big) \right| \right) \leqq$$

$$\leqq \sum_{m=1}^{m_0} \frac{1}{m} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \sum_{j=1}^{P} 2\pi \left| \frac{mt}{n} \sum_{l=k+2}^{+\infty} (-1)^l \left(\frac{j}{n}\right)^l \right| \right) \leqq$$

$$\leqq \sum_{m=1}^{m_0} \frac{1}{m} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \sum_{j=1}^{P} 2\pi \frac{mt}{n} \left(\frac{j}{n}\right)^{k+2} \right) \leqq$$

$$\leqq \sum_{m=1}^{m_0} \left(\left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + 2\pi t \frac{P^{k+3}}{n^{k+3}} \right) = \sum_{m=1}^{m_0} \left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + 2\pi m_0 t \frac{P^{k+3}}{n^{k+3}} <$$

$$< \sum_{m=1}^{m_0} \left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + 2\pi \frac{49}{\varepsilon^2} t \frac{P^{k+3}}{n^{k+3}} < \sum_{m=1}^{m_0} \left| \sum_{j=1}^{P} e\big(f_m(j)\big) \right| + \frac{400}{\varepsilon^2} t \frac{P^{k+3}}{n^{k+3}}.$$

Now we are going to estimate the parameters k, P. Firs we note that in this case, (4) can be rewritten in the form

$$k \leqq \frac{\log t}{\log n} + \frac{1}{2} < k+1, \tag{30}$$

i.e.,

$$n^{k-1/2} \leqq t < n^{k+1/2}. \tag{31}$$

Furthermore, with respect to (31) we have

$$P = \left[\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] \leqq \left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)} \leqq \left(\frac{n^{k+5/2}}{n^{k-1/2}}\right)^{1/(k+2)} = n^{3/(k+2)} \tag{32}$$

and

$$P = \left[\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] > \left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)} - 1 > \tag{33}$$

$$> \left(\frac{n^{k+5/2}}{n^{k+1/2}}\right)^{1/(k+2)} - 1 = n^{2/(k+2)} - 1 > \frac{1}{2} n^{2/(k+2)}$$

(note that $n^{2/(k+2)} \to +\infty$ follows easily from (3) and (30)).

For large n, the last term in (29) can be estimated in the following way:

$$\frac{400}{\varepsilon^2} t \frac{P^{k+3}}{n^{k+3}} = \frac{400}{\varepsilon^2} t \left[\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right]^{k+2} \frac{P}{n^{k+3}} \leqq \tag{34}$$

$$\leqq \frac{400}{\varepsilon^2} t \frac{n^{k+5/2}}{t} \frac{P}{n^{k+3}} = \frac{400}{\varepsilon^2} \frac{1}{n^{1/2}} P < \frac{\varepsilon}{32} P.$$

Finally, we estimate the sum $\left|\sum_{j=1}^{P} e(f_m(j))\right|$ by using Lemma 3 with 0 and $f_m(x)$ in place of Q and $f(x)$, respectively. In fact, by (28) and (30) we have

$$k > \frac{\log t}{\log n} - \frac{1}{2} \geqq c_1 - \frac{1}{2} \geqq \left(c_3 + \frac{1}{2}\right) - \frac{1}{2} = c_3$$

so that (14) holds. Furthermore, with respect to (30) we have

$$|\alpha| = \left|(-1)^{k+1} \frac{mt}{n^{k+2}}\right| = m \frac{t}{n^{k+2}} = mn^{\frac{\log t}{\log n} - k - 2} \leqq \tag{35}$$

$$\leqq m_0 n^{\left(\frac{\log t}{\log n} + \frac{1}{2}\right) - (k+1) - \frac{3}{2}} < \frac{49}{\varepsilon^2} n^{-3/2}.$$

(3), (30), (32) and (35) yield for large n that

$$0 < 2(k+1)P|\alpha| < 2\left(\frac{\log t}{\log n} + \frac{3}{2}\right) n^{3/(k+2)} \frac{49}{\varepsilon^2} n^{-3/2} <$$

$$< 2\left(\frac{1}{\log n} \frac{(\log n)^{5/4}}{\log\log\log n} + \frac{3}{2}\right) n \frac{49}{\varepsilon^2} n^{-3/2} < (\log n)^{1/4} n^{-1/2} < 1$$

so that also (15) holds. Thus Lemma 2 can be applied, and we obtain that

$$\left|\sum_{j=1}^{P} e\big(f_m(j)\big)\right| \leqq 2e^{15k\log^2 k} P^{1-1/6k^2\log k}\log P + 2|\alpha|^{-1/k}. \tag{36}$$

First we estimate the first term on the right hand side. By (3), (30), (32) and (33), for large n we have

$$2e^{15k\log^2 k} P^{1-1/6k^2\log k}\log P = \tag{37}$$

$$= 2P\exp\left(15k\log^2 k - \frac{\log P}{6k^2\log k} + \log\log P\right) <$$

$$< 2P\exp\left(15\left(\frac{\log t}{\log n}+\frac{1}{2}\right)\log^2\left(\frac{\log t}{\log n}+\frac{1}{2}\right) - \frac{\log\frac{1}{2}n^{2/(k+2)}}{6k^2\log k} + \log\log n^{3/(k+2)}\right) <$$

$$< 2P\exp\left(30\frac{\log t}{\log n}\log^2\left(\frac{\log t}{\log n}\right) - \frac{\log n}{6k^2(k+2)\log k} + \log\log n\right) <$$

$$< 2P\exp\left(30\frac{1}{\log n}\frac{(\log n)^{5/4}}{\log\log n}\log^2\left(\frac{1}{\log n}\frac{(\log n)^{5/4}}{\log\log n}\right) - \frac{\log n}{18k^3\log k} + \log\log n\right) <$$

$$< 2P\exp\left(30\frac{(\log n)^{1/4}}{\log\log n}(\log\log n)^2 - \frac{\log n}{18\left(\frac{\log t}{\log n}+\frac{1}{2}\right)^3\log\left(\frac{\log t}{\log n}+\frac{1}{2}\right)} + \log\log n\right) <$$

$$< 2P\exp\left(31(\log n)^{1/4}\log\log n - \frac{\log n}{100\left(\frac{\log t}{\log n}\right)^3\log\left(\frac{\log t}{\log n}\right)}\right) <$$

$$< 2P\exp\left(31(\log n)^{1/4}\log\log n - \frac{\log n}{100\left(\frac{(\log n)^{1/4}}{\log\log n}\right)^3\log\frac{(\log n)^{1/4}}{\log\log n}}\right) <$$

$$< 2P\exp\left(31(\log n)^{1/4}\log\log n - \frac{\log n}{100\frac{(\log n)^{3/4}}{(\log\log n)^3}\log\log n}\right) =$$

$$= 2P\exp\left(31(\log n)^{1/4}\log\log n - \frac{1}{100}(\log n)^{1/4}(\log\log n)^2\right) <$$

$$< 2P\exp\left(-\frac{1}{101}(\log n)^{1/4}(\log\log n)^2\right) < P\exp\left(-(\log n)^{1/5}\right).$$

With respect to $k \geqq c_1 \geqq 20$, (3), (28), (30), (31), (33) and (35), for large n the second term on the right hand side of (36) can be estimated in the following way:

$$(38) \qquad 2|\alpha|^{-1/k} = 2\left(\frac{mt}{n^{k+2}}\right)^{-1/k} = 2\left(\frac{n^{k+2}}{mt}\right)^{1/k} \leqq 2\left(\frac{n^{k+2}}{t}\right)^{1/k} =$$

$$= 2\left(\frac{n^{k+5/2}}{t}\right)^{1/k} n^{-1/2k} = 2\left(\frac{n^{k+5/2}}{t}\right)^{1/(k+2)} \left(\frac{n^{k+5/2}}{t}\right)^{1/k-1/(k+2)} n^{-1/2k} \leqq$$

$$\leqq 4P\left(\frac{n^{k+5/2}}{t}\right)^{2/k(k+2)} n^{-1/2k} \leqq 4P\left(\frac{n^{k+5/2}}{n^{k-1/2}}\right)^{2/k(k+2)} n^{-1/2k} =$$

$$= 4Pn^{6/k(k+2)-1/2k} = 4Pn^{(10-k)/2k(k+2)} \leqq$$

$$\leqq 4Pn^{(k/2-k)/2k(k+2)} = 4Pn^{-1/4(k+2)} < 4Pn^{-1/12k} =$$

$$= 4P\exp\left(-\frac{\log n}{12k}\right) \leqq 4P\exp\left(-\frac{\log n}{12\left(\frac{\log t}{\log n}+\frac{1}{2}\right)}\right) <$$

$$< 4P\exp\left(-\frac{\log n}{12\left(\frac{(\log n)^{1/4}}{\log\log n}+\frac{1}{2}\right)}\right) < P\exp\left(-\frac{\log n}{(\log n)^{1/4}}\right) = P\exp\left(-(\log n)^{3/4}\right).$$

(29), (34), (36), (37) and (38) yield for large n that

$$\sum_{m=1}^{m_0}\frac{1}{m}\left|\sum_{j=1}^{P} e\left(m\frac{t}{n+j}\right)\right| < \sum_{m=1}^{m_0}\left|\sum_{j=1}^{P} e(f_m(j))\right| + \frac{400}{\varepsilon^2}\, t\,\frac{P^{k+3}}{n^{k+3}} <$$

$$< \sum_{m=1}^{m_0}\left(P\exp\left(-(\log n)^{1/5}\right) + P\exp\left(-(\log n)^{3/4}\right)\right) + \frac{\varepsilon}{32}P <$$

$$< 2m_0 P\exp\left(-(\log n)^{1/5}\right) + \frac{\varepsilon}{32}P =$$

$$= 2\left(\left[\frac{48}{\varepsilon^2}\right]+1\right)P\exp\left(-(\log n)^{1/5}\right) + \frac{\varepsilon}{32}P < \frac{\varepsilon}{32}P + \frac{\varepsilon}{32}P = \frac{\varepsilon}{16}P$$

which proves that (27) holds also in Case 2.

(Note that like Case 1, also Case 2 could be treated in a simpler way by replacing Lemma 3 by Theorem 1 in [5], p. 47; in fact in this way we can show that the exponent 5/4 in the upper bound in (3) can be replaced by the greater on 3/2, but, on the other hand, this methods yields the much worse estimate $P \sim n^{1-c(\log t/\log n)^{-2}} \sim n^{1-c/k^2}$ for P; this is why we have preferred the more complicated way based on Lemma 3.)

We obtain from (26) and (27) that

$$N(\eta, \varrho) < \left(\varrho-\eta+\frac{3\varepsilon}{16}\right)P + \frac{\varepsilon}{16}P = \left(\varrho-\eta+\frac{\varepsilon}{4}\right)P$$

provided that (17) holds:

$$N(\eta, \varrho) < \left(\varrho-\eta+\frac{\varepsilon}{4}\right) P \quad \text{for} \quad \frac{\varepsilon}{4} \leqq \varrho-\eta \leqq 1-\frac{\varepsilon}{4}. \tag{39}$$

Assume now that $0 \leqq \varrho-\eta < \varepsilon/4$. Then (39) yields (with $\eta+\frac{\varepsilon}{4}$ in place of ϱ) that

$$N(\eta, \varrho) \leqq N\left(\eta, \varrho+\left(\frac{\varepsilon}{4}-(\varrho-\eta)\right)\right) = N\left(\eta, \eta+\frac{\varepsilon}{4}\right) < \tag{40}$$

$$< \left(\left(\eta+\frac{\varepsilon}{4}\right)-\eta+\frac{\varepsilon}{4}\right) P = \frac{\varepsilon}{2} P \leqq \left(\varrho-\eta+\frac{\varepsilon}{2}\right) P \quad \text{for} \quad 0 \leqq \varrho-\eta < \varepsilon/4.$$

Finally, let $1-\frac{\varepsilon}{4} < \varrho-\eta = 1$. Then we have

$$N(\eta, \varrho) \leqq P = \left(\left(1-\frac{\varepsilon}{4}\right)+\frac{\varepsilon}{4}\right) P < \left(\varrho-\eta+\frac{\varepsilon}{4}\right) P \quad \text{for} \quad 1-\frac{\varepsilon}{4} < \varrho-\eta \leqq 1. \tag{41}$$

(39), (40) and (41) yield that

$$N(\eta, \varrho) < \left(\varrho-\eta+\frac{\varepsilon}{2}\right) P \quad \text{for all} \quad 0 \leqq \varrho-\eta \leqq 1. \tag{42}$$

On the other hand, by using (42) repeatedly, we obtain that

$$N(\alpha, \beta) = N(0, 1)-N(0, \alpha)-N(\beta, 1) = P-N(0, \alpha)-N(\beta, 1) > \tag{43}$$

$$> P-\left(\alpha+\frac{\varepsilon}{2}\right) P-\left(1-\beta+\frac{\varepsilon}{2}\right) P = (\beta-\alpha-\varepsilon) P \quad \text{for all} \quad 0 \leqq \alpha < \beta \leqq 1.$$

(42) and (43) yield (7) and this completes the proof of Theorem 1.

3. In this section, we prove the following consequence of Theorem 1:

THEOREM 2. *Let $\varepsilon > 0$, n a positive integer satisfying $n > n_1(\varepsilon)$, t a real number such that*

$$n < t < \exp\left(\frac{(\log n)^{5/4}}{\log \log n}\right).$$

Let us define k by (4) (where c_1 denotes the constant defined in Theorem 1), and write

$$P = \begin{cases} n \text{ if } n < t < n^2 \\ [n^{1-1/2^{k+2}}] \text{ if } n^2 \leqq t < n^{c_1} \\ \left[\left(\dfrac{n^{k+5/2}}{t}\right)^{1/(k+2)}\right] \text{ if } n^{c_1} \leqq t. \end{cases}$$

Then there exists a positive integer j such that

$$(44) \qquad 1 \leqq j \leqq P$$

and

$$(45) \qquad (n+j)|_\varepsilon t.$$

PROOF. We have to distinguish three cases.

Case 1. Let

$$(46) \qquad n < t < \varepsilon n^2.$$

If $n<t<2n+2$, then (45) holds with

$$j = \begin{cases} 1 \text{ for } n<t<n+1 \\ [t-n] \text{ for } n+1 \leqq t < 2n+1 \\ n \text{ for } 2n+1 \leqq t < 2n+2 \end{cases}$$

(for large n). Thus we may assume that $2n+2 \leqq t$; hence

$$(47) \qquad \left[\frac{t}{n+1}\right] \geqq 2.$$

Let us write t in the form

$$(48) \qquad t = \left[\frac{t}{n+1}\right](n+1)+r \quad \text{where} \quad 0 \leqq r < n+1$$

and

$$(49) \qquad t = \left[\frac{t}{2n}\right](2n)+s \quad \text{where} \quad 0 \leqq s < 2n,$$

respectively. (48) and (49) yield that

$$(50) \qquad 2\left(\left[\frac{t}{n+1}\right]-\left[\frac{t}{2n}\right]\right)n = \left[\frac{t}{n+1}\right](n-1)-r+s.$$

By (47), (48) and (49), we have

$$(51) \qquad \left[\frac{t}{n+1}\right](n-1)-r+s \geqq 2(n-1)-r > 2(n-1)-(n+1) = n-3 > 0$$

for $n>3$. (50) and (51) yield (for $n>3$) that

$$\left[\frac{t}{n+1}\right] > \left[\frac{t}{2n}\right].$$

Thus there exists an integer j such that $1 \leqq j \leqq n-1$ $(=P-1)$ and

$$(52) \qquad \left[\frac{t}{n+j}\right] > \left[\frac{t}{n+j+1}\right].$$

We are going to show that this integer j satifies also (45).

Let us write $q=\left[\frac{t}{n+j}\right]$. Then by (52), we have $\frac{t}{n+j} \geqq q > \frac{t}{n+j+1}$, thus with respect to (46)

$$0 \leqq \frac{t}{n+j} - q < \frac{t}{n+j} - \frac{t}{n+j+1} = \frac{t}{(n+j)(n+j+1)} < \frac{t}{n^2} < \varepsilon$$

which implies (45) and this completes the proof of the theorem in this case.

Case 2. Let

(53) $$\varepsilon n^2 \leqq t < n^2.$$

For $j=1, 2, \ldots, n-1$, let

$$d_j = \frac{t}{n+j} - \frac{t}{n+j+1} = \frac{t}{(n+j)(n+j+1)}.$$

Then obviously,

(54) $$0 < d_{n-1} < d_{n-2} < \ldots < d_2 < d_1.$$

By (53), for sufficiently large n we have

(55) $$d_1 - d_{n-[n/3]} = \frac{t}{(n+1)(n+2)} - \frac{t}{(2n-[n/3])(2n-[n/3]+1)} >$$

$$> \frac{t}{\left(\frac{4}{3}n\right)^2} - \frac{t}{\left(\frac{5}{3}n\right)^2} = \frac{81}{400}\frac{t}{n^2} > \frac{1}{5}\frac{t}{n^2} \geqq \frac{\varepsilon}{5}.$$

Let p denote number such that

(56) $$\frac{10}{\varepsilon} < p < \frac{20}{\varepsilon}.$$

(It is well-known that for $x \geqq 2$, there exists a prime number q such that $x<q<2x$.) (56) yields that

(57) $$\frac{1}{p} < \frac{\varepsilon}{10}.$$

(55) and (57) imply that there exists an integer a such that

(58) $$d_{n-[n/3]} < \frac{a}{p} < \frac{a+1}{p} < d_1.$$

Then either

(59) $$(a, p) = 1$$

or $(a+1, p)=1$ holds; we may assume that (59).

(54) and (58) imply that there exists an integer u such that

(60) $$1 \leqq u \leqq n-[n/3]-1 < \frac{2n}{3}$$

and

(61) $$d_{u+1} \leqq \frac{a}{p} < d_u.$$

By (53), for $j=1, 2, \ldots, n-2$ we have

(62) $$0 < d_j - d_{j+1} = \frac{t}{(n+j)(n+j+1)} - \frac{t}{(n+j+1)(n+j+2)} =$$

$$= \frac{2t}{(n+j)(n+j+1)(n+j+2)} < \frac{2t}{n^3} < \frac{2}{n} \quad (\text{for } j = 1, 2, \ldots, n-2).$$

(61) and (62) yield that

$$0 < d_u - \frac{a}{p} \leqq d_u - d_{u+1} < \frac{2}{n}$$

hence

(63) $$\left|d_u - \frac{a}{p}\right| < \frac{2}{n}.$$

Obviously, there exists an integer b such that

(64) $$\left|\frac{t}{n+u} - \frac{b}{p}\right| < \frac{1}{2p}.$$

Define the integer l by

(65) $$al \equiv b \pmod{p}$$

(such an l exists by (59)) and

(66) $$1 \leqq l \leqq p.$$

Put $q = \dfrac{b-al}{p}$ and $j=u+l$. (56,) (60) and (66) yield for large n that

(67) $$(1 \leqq) j = u + l < \frac{2n}{3} + p < \frac{2n}{3} + \frac{20}{\varepsilon} < \frac{2n}{3} + \frac{n}{3} = n.$$

For $i=1, 2, \ldots, n-1-u$, we have

$$d_{u+i} = d_u + (d_{u+1} - d_u) + (d_{u+2} - d_{u+1}) + \ldots + (d_{u+i} - d_{u+i-1})$$

thus by (62) and (63),

(68) $$\left|d_{u+i} - \frac{a}{p}\right| \leqq |d_{u+i} - d_u| + \left|d_u - \frac{a}{p}\right| \leqq$$

$$\leqq |d_{u+1} - d_u| + |d_{u+2} - d_{u+1}| + \ldots + |d_{u+i} - d_{u+i-1}| + \frac{2}{n} < i\frac{2}{n} + \frac{2}{n} = \frac{2(i+1)}{n}$$

$$(\text{for } 0 \leqq i \leqq n-1-u).$$

Furthermore, we have

$$\frac{t}{n+j}=\frac{t}{n+u+l}=\frac{t}{n+u}-\left(\frac{t}{n+u}-\frac{t}{n+u+1}\right)-\left(\frac{t}{n+u+1}-\frac{t}{n+u+2}-\dots-\right.$$

$$\left.-\left(\frac{t}{n+u+l-1}-\frac{t}{n+u+l}\right)=\frac{t}{n+u}-d_u-d_{u+1}-\dots-d_{u+l-1}=\right.$$

$$=\left(\frac{t}{n+u}-\frac{b}{p}\right)+\frac{b-la}{p}-\sum_{i=0}^{l-1}\left(d_{u+i}-\frac{a}{p}\right)=\left(\frac{t}{n+u}-\frac{b}{p}\right)+q-\sum_{i=0}^{l-1}\left(d_{u+i}-\frac{a}{p}\right)$$

thus with respect to (56), (57), (64), (66) and (68)

$$\left|\frac{t}{n+j}-q\right|\leqq\left|\frac{t}{n+u}-\frac{b}{p}\right|+\sum_{i=0}^{l-1}\left|d_{u+i}-\frac{a}{p}\right|<\frac{1}{2p}+\sum_{i=0}^{l-1}\frac{2(i+1)}{n}\leqq$$

$$\leqq\frac{1}{2p}+\frac{2l^2}{n}\leqq\frac{1}{2p}+\frac{2p^2}{n}<\frac{\varepsilon}{20}+\frac{800}{\varepsilon^2 n}<\frac{\varepsilon}{20}+\frac{\varepsilon}{2}<\varepsilon$$

which implies that

$$\left\|\frac{t}{n+j}\right\|<\varepsilon. \tag{69}$$

(67) and (69) show that (44) and (45) hold also in Case 2.

Case 3. Let

$$n^2\leqq t<\exp\left(\frac{(\log n)^{5/4}}{\log\log n}\right).$$

Then by using Theorem 1 with $\frac{\varepsilon}{2}$ in place of ε, we obtain for large n that

$$\sum_{\substack{1\leqq j\leqq P\\ (n+j)|_\varepsilon t}}1=\sum_{\substack{1\leqq j\leqq P\\ \left\|\frac{t}{n+j}\right\|<\varepsilon}}1\geqq\sum_{\substack{1\leqq j\leqq P\\ 0\leqq\left\{\frac{t}{n+j}\right\}<\varepsilon}}1=N(0,\varepsilon)=$$

$$=\varepsilon P+\bigl(N(0,\varepsilon)-\varepsilon P\bigr)\geqq\varepsilon P-|N(0,\varepsilon)-\varepsilon P|>\varepsilon P-\frac{\varepsilon}{2}P=\frac{\varepsilon}{2}P>1$$

which shows that there exists an integer j satifying (44) and (45), and this completes the proof of Theorem 2.

4. In this section, we show that if t is large (in terms of n) then it may occur that there does not exist an integer j satisfying $1\leqq j\leqq n$ and $(n+j)|_\varepsilon t$.

THEOREM 3. *Let* $\frac{1}{4}>\varepsilon>0$, $\delta>0$. *Then for* $n>n_2(\varepsilon)$, *there exists a real number* t *such that*

$$n<t<\exp\bigl((2+\delta)n\bigr) \tag{70}$$

and there does not exist an integer j satisfying $1 \leqq j \leqq n$ *and*

$$(n+j)\,|_{\varepsilon}\, t. \tag{71}$$

PROOF. Let $t = [1, 2, \dots, 2n] + \frac{n}{2}$ (where $[1, 2, \dots, 2n]$ denotes the least common multiple of the numbers $1, 2, \dots, 2n$); then $n < t$ holds trivially. For $p \leqq 2n$, define the positive integer α_p by

$$p^{\alpha_p} \leqq 2n < p^{\alpha_p + 1}.$$

Then by the prime number theorem, we have

$$\log [1, 2, \dots, 2n] = \log \Big(\prod_{p \leqq 2n} p^{\alpha_p} \Big) = \sum_{p \leqq 2n} \log p^{\alpha_p} \sim 2n$$

so that for large n,

$$t = [1, 2, \dots, 2n] + \frac{n}{2} < \exp \left(\left(2 + \frac{\delta}{2} \right) n \right) + \frac{n}{2} < \exp \left((2+\delta) n \right)$$

which proves (70).

Furthermore, if $1 \leqq j \leqq n$ then

$$\left\{ \frac{t}{n+j} \right\} = \left\{ \frac{[1, 2, \dots, 2n] + n/2}{n+j} \right\} = \left\{ \frac{[1, 2, \dots, 2n]}{n+j} + \frac{n}{2(n+j)} \right\} = \left\{ \frac{n}{2(n+j)} \right\}.$$

Here we have

$$\frac{1}{4} = \frac{n}{4n} \leqq \frac{n}{2(n+j)} < \frac{n}{2n} = \frac{1}{2}$$

hence

$$\frac{1}{4} \leqq \left\{ \frac{t}{n+j} \right\} = \left\{ \frac{n}{2(n+j)} \right\} = \frac{n}{2(n+j)} < \frac{1}{2}$$

which implies that

$$\frac{1}{4} \leqq \left\| \frac{t}{n+j} \right\| = \left\{ \frac{t}{n+j} \right\}.$$

Thus (71) does not hold which completes the proof of Theorem 3.

5. Note that there is a considerable gap between Theorems 2 and 3. In fact, let $f(n, \varepsilon)$ denote the infimum of the real numbers t such that $n < t$ and there does not exist an integer j such that $1 \leqq j \leqq n$ and $(n+j)\,|_{\varepsilon}\, t$. Then for $n > n_0(\varepsilon)$, Theorem 2 shows that

$$\exp \left(\frac{(\log n)^{5/4}}{\log \log n} \right) \leqq f(n, \varepsilon) \tag{72}$$

and on the other hand, Theorem 3 yields that

$$f(n, \varepsilon) \leqq \exp \left((2+\delta) n \right); \tag{73}$$

we guess that both the lower estimate (72) and the upper estimate (73) are far from the best possible.

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