Studies in Pure Mathematics
To the Memory of Paul Turán

On some problems of J. Dénes and P. Turán

by

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1. In what follows we are dealing with some statistical properties of partitions resp. unequal partitions of positive integers. We introduce the notation

$$\Pi = \left\{ \begin{matrix} \lambda_1 + \lambda_2 + \ldots + \lambda_m = n \\ \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m \geqq 1 \end{matrix} \right\} \tag{1.1}$$

for a generic partition Π of n where

$$m = m(\Pi) \quad \text{and the} \quad \lambda_\mu\text{'s are integers.} \tag{1.2}$$

Let $p(n)$ denote the number of partitions of n. According to the classical result of G. H. Hardy and S. Ramanujan (see [1]),

$$p(n) = (1 + o(1)) \frac{1}{4n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right). \tag{1.3}$$

(The o-sign and later the O-sign refer to $n \to \infty$.)

2. J. Dénes raised the following interesting problem. What is the number of pairs (Π_1, Π_2) of partitions of n which do not have equal *subsums*? This problem has not been solved yet but its investigation led P. Turán to some unexpected phenomena. The pairs with the Dénes property* are obviously contained in the set of pairs of partitions not having common *summands*. P. Turán proved (see [6]) that the number of pairs of partitions (of n) having no common summands is

$$\exp((1 + o(1))\pi\sqrt{2n}) \tag{2.1}$$

* Apart from the common complete subsums of course, we exclude the pair ($\lambda_1 = n$, $\lambda_1 = n$) here.

at most. This estimation shows that the number of pairs with the Dénes property is "small" (in comparison with the total number $p(n)^2$ of the pairs). This smallness suggested that "almost all" pairs (i.e., with the exception of $o(p(n)^2)$ pairs at most) have "many" common summands. Indeed, P. TURÁN proved (see [6]) that almost all pairs of partitions of n contain

$$\left(\frac{\sqrt{6}}{4\pi} - o(1)\right)\sqrt{n}\log n \tag{2.2}$$

common summands at least (*with multiplicity*). Afterwards P. TURÁN proved an analogue of the above result for k-tuples of partitions with fixed integer $k \geqq 2$ (see [7]). This result was generalized for $k = o(\sqrt{n})$ by C. POMERANCE [2].

Thinking of the fact (which is easy to prove) that "almost all" partitions of n (i.e., with the exception of $o(p(n))$ partitions at most) contain 1 as summand $[\sqrt{n}(\omega(n))^{-1}]$-times at least ($\omega(n) \nearrow \infty$ arbitrarily slowly) one can imagine that the phenomenon (2.2) is perhaps caused by certain summands of *great multiplicity*. That this is *not* the "real" reason turned out in [8]. Namely, in his paper [8] P. TURÁN proved the existence of

$$(1 - o(1))\frac{\sqrt{3}}{\pi k 2^{k-1}}\sqrt{n} \tag{2.3}$$

common summands in "almost all" k-tuples of *unequal partitions* of the form

$$\Pi^* = \begin{Bmatrix} \alpha_1 + \alpha_2 + \ldots + \alpha_m = n \\ \alpha_1 > \alpha_2 > \ldots > \alpha_m \geqq 1 \end{Bmatrix} \tag{2.4}$$

where

$$m = m(\Pi^*) \quad \text{and the} \quad \alpha_\mu\text{'s are integers.} \tag{2.5}$$

We remind the reader that G. H. HARDY and S. RAMANUJAN's formula (see [1]) asserts the relation

$$q(n) = \frac{1 + o(1)}{4n^{3/4}3^{1/4}} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right) \tag{2.6}$$

for the number $q(n)$ of *unequal partitions* (2.4)–(2.5) of n.

3. Another approach to the original problem would be, as P. TURÁN proposed, the investigation of the integers which can be represented by subsums. This investigation led us to other surprising phenomena we are dealing with in this paper. Our Theorem I yields that (not in the strongest form) *almost all* partitions of n represent *all* integers k of $[1, n]$ as subsums, i.e., in the form

$$k = \sum_{j=1}^{S_k} \lambda_{i_j} \qquad (i_j \neq i_l \quad \text{for} \quad j \neq l). \tag{3.1}$$

The analogue of this assertion does not hold for *unequal partitions* (e.g., it is easy to see that $k=1$ cannot be represented in a positive percentage of the unequal partitions of n) but our Theorem II yields a weaker result of similar type.

4. Let $M(n)$ denote the number of such partitions Π of n for which it is *not* true that every integer k of the interval $[1, n]$ is representable by a subsum of Π. Then we assert

Theorem I.

$$M(n) = \left(1 + O\left(\frac{\log^{30} n}{\sqrt{n}}\right)\right)\frac{\pi}{\sqrt{6n}}\, p(n)\,. \tag{4.1}$$

Corollary. *The number of partitions of n which represent all integers k of the interval $[1, n]$ as subsums is*

$$\left(1 - \frac{\pi}{\sqrt{6n}} + O\left(\frac{\log^{30} n}{n}\right)\right)p(n)\,, \tag{4.2}$$

consequently, almost all partitions of n represent all integers k of $[1, n]$ as subsums.

For the proof of Theorem I, we need a number of lemmata. We use the results of P. Turán and M. Szalay on the distribution of summands in the partitions of n (see [3], [4], [5]).

5. Using the notation (1.1), we define

$$S_1(n, \Pi, \Lambda) = \sum_{\substack{\lambda_\nu \geqq \Lambda \\ \lambda_\nu \in \Pi \text{ (with multiplicity)}}} 1. \tag{5.1}$$

Lemma 1 (M. Szalay–P. Turán [3], Corollary of Theorem II). *If Λ is restricted by*

$$11 \log n \leqq \Lambda \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 3\sqrt{n}\log\log n \tag{5.2}$$

then the relation

$$S_1(n, \Pi, \Lambda) = \left(1 + O\left(\frac{1}{\log n}\right)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\Lambda}{\sqrt{6n}}\right)} \tag{5.3}$$

holds uniformly in (5.2) apart from

$$c\,p(n)\, n^{-5/4}\log n \tag{5.4}$$

exceptional Π's at most.

Throughout this paper c's stand for explicitly calculable positive constants not necessarily the same in different occurrences.

Lemma 2 (M. SZALAY–P. TURÁN [5], Corollary 1). *With the restriction*

$$\log^6 n \leqq \mu \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 5\sqrt{n}\log\log n \tag{5.5}$$

the relation

$$\lambda_\mu = \left(1 + O\left(\frac{1}{\log n}\right)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)} \tag{5.6}$$

holds uniformly with the exception of $cp(n)\,n^{-5/4}\log n$ *partitions of* n *at most.*

Lemma 3 (M. SZALAY–P. TURÁN [5], Lemma 4). *The inequalities*

$$\lambda_1 \leqq 5\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n \tag{5.7}$$

and

$$m \leqq 5\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n \tag{5.8}$$

hold with the exception of $cp(n)\,n^{-2}$ *Π's at most.*

Lemma 4. *Using the abbreviation*

$$U(k) = \log\frac{1}{1-\exp\left(-\frac{\pi k}{\sqrt{6n}}\right)}, \tag{5.9}$$

we have, for

$$\log^7 n \leqq k \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 9\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n, \tag{5.10}$$

the uniform estimation

$$\sum_{\mu=k}^{m}\lambda_\mu = \left(1 + O\left(\frac{1}{\log n}\right)\right)\frac{6}{\pi^2}n\int_0^{U(k)}\frac{x}{\exp(x)-1}\,dx \tag{5.11}$$

apart from $cp(n)\,n^{-5/4}\log n$ *Π's at most.*

Proof. Owing to Lemma 2 and Lemma 3, we have

$$\sum_{\mu=k}^{m}\lambda_\mu = \sum_{\mu\in I}\lambda_\mu + O(m)\,O(\log^7 n)$$

where

$$I = \left[k, \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 7\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n \right],$$

i.e.,

$$\sum_{\mu=k}^{m} \lambda_\mu = \sum_{\mu\in I} \lambda_\mu + O(\sqrt{n}\log^8 n) =$$

$$= \sum_{\substack{\mu\in I\\ \mu\,\text{integer}}} \left(1 + O\left(\frac{1}{\log n}\right)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)} + O(\sqrt{n}\log^8 n) =$$

$$= \left(1 + O\left(\frac{1}{\log n}\right)\right)\left\{ \int_I \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi x}{\sqrt{6n}}\right)}dx + O(\sqrt{n}\log n)\right\} +$$

$$+ O(\sqrt{n}\log^8 n) =$$

$$= \left(1 + O\left(\frac{1}{\log n}\right)\right)\cdot$$

$$\cdot\left\{ \int_k^\infty \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi x}{\sqrt{6n}}\right)}dx + O(\sqrt{n}\log^7 n)\right\} + O(\sqrt{n}\log^8 n).$$

Here, the last integral is

$$\geqq \frac{6}{\pi^2}\sqrt{n}\log^9 n,$$

thus,

$$\sum_{\mu=k}^{m} \lambda_\mu = \left(1 + O\left(\frac{1}{\log n}\right)\right)\int_k^\infty \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi x}{\sqrt{6n}}\right)}dx =$$

$$= \left(1 + O\left(\frac{1}{\log n}\right)\right)\frac{6}{\pi^2}n\int_0^{U(k)} \frac{y}{\exp(y)-1}dy$$

$$\left(\text{with } y = \log\frac{1}{1-\exp\left(-\frac{\pi x}{\sqrt{6n}}\right)}\right).$$

Lemma 5. *There exists a positive constant c such that dropping* $cp(n)n^{-5/4}\log n$ *suitable partitions of n at most, the remaining ones have the property that*

$$\lambda_\mu \geqq \log^{10} n > 0 \tag{5.12}$$

implies

$$\lambda_\mu \leqq \lambda_{\mu+1} + \lambda_{\mu+2} + \ldots + \lambda_m . \tag{5.13}$$

Proof. After dropping $cp(n)\,n^{-5/4}\log n$ Π's at most all the previous lemmata will be applicable. Owing to Lemma 2,

$$\lambda_{\left[\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 9\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n\right]} < c\log^{9.1} n < \log^{10} n$$

for $n > n_0$. Therefore, for sufficiently large n,

$$\lambda_\mu \geqq \log^{10} n$$

implies

$$1 \leqq \mu \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 9\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n - 1 . \tag{5.14}$$

For

$$n > n_1 \quad \text{and} \quad 1 \leqq \mu \leqq \left[\frac{\sqrt{6}}{\pi}\sqrt{n}\right] - 1 , \tag{5.15}$$

Lemma 4 and Lemma 3 yield that

$$\lambda_{\mu+1} + \lambda_{\mu+2} + \ldots + \lambda_m \geqq \lambda_{\left[\frac{\sqrt{6}}{\pi}\sqrt{n}\right]} + \ldots + \lambda_m \geqq$$

$$\geqq \frac{3}{\pi^2} n \int_0^{\log\frac{1}{1-\exp(-1)}} \frac{x}{\exp(x)-1}\,dx > 5\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n \geqq \lambda_1 \geqq \lambda_\mu ,$$

thus the inequality (5.13) holds in the case (5.15). Next let

$$n > n_2, \quad \left[\frac{\sqrt{6}}{\pi}\sqrt{n}\right] \leqq \mu \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 9\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n - 1 . \tag{5.16}$$

Then, owing to Lemma 4 resp. Lemma 2, we get

$$\lambda_{\mu+1} + \lambda_{\mu+2} + \ldots + \lambda_m \geqq \frac{3}{\pi^2} n \int_0^{\exp\left(-\frac{\pi(\mu+1)}{\sqrt{6n}}\right)} \frac{x}{\exp(x)-1}\,dx \geqq$$

$$\geqq \frac{3}{\pi^2} n \int\limits_0^{\exp\left(-\frac{\pi(\mu+1)}{\sqrt{6n}}\right)} \frac{1}{e-1}\, dx > \frac{3}{2\pi^2} n \exp\left(-\frac{\pi(\mu+1)}{\sqrt{6n}}\right) > \frac{1}{100} n \exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right).$$

resp.

$$\lambda_\mu \leqq c\sqrt{n} \exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right) < \frac{1}{100} n \exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right).$$

These estimations imply (5.13) in the case (5.16). Thus, Lemma 5 is proved for sufficiently large n and the increase of the constant c completes the proof for all n.

6. We shall use HARDY–RAMANUJAN's stronger formula (see [1]) in the form

$$(6.1)\quad p(n) = \frac{\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n-\frac{1}{24}}\right)}{4\left(n-\frac{1}{24}\right)\sqrt{3}} \left\{1 - \frac{\sqrt{6}}{2\pi\sqrt{n-\frac{1}{24}}}\right\} + O\left(\exp\left(0.51 \cdot \frac{2\pi}{\sqrt{6}}\sqrt{n}\right)\right).$$

One can get easily from (6.1) that

$$(6.2)\qquad p(n) = \frac{1}{4n\sqrt{3}} \left\{1 - \left(\frac{\sqrt{6}}{2\pi} + \frac{\pi}{24\sqrt{6}}\right)\frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right\} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right).$$

Let $p_1(n)$ denote the number of partitions of n not containing 1 as summand. We have obviously

$$(6.3)\qquad p_1(n) = p(n) - p(n-1) \quad \text{for} \quad n > 1$$

and using (6.2) we get

$$p_1(n) = p(n)\left(1 - \frac{p(n-1)}{p(n)}\right) = p(n)\left(1 - \left(1 + O\left(\frac{1}{n}\right)\right)\exp\left(-\frac{2\pi}{\sqrt{6}}\frac{1}{\sqrt{n}+\sqrt{n-1}}\right)\right) =$$

$$= p(n)\left(\frac{\pi}{\sqrt{6n}} + O\left(\frac{1}{n}\right)\right).$$

Thus we have proved

Lemma 6.

$$(6.4)\qquad p_1(n) = p(n) - p(n-1) = \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)\frac{\pi}{\sqrt{6n}} p(n)$$

for $n > 1$.

To the representations of the "small" integers we shall need the number $p(n, i, j)$ of partitions of n containing neither the summand i nor the summand j where i, j are integers and

$$1 \leqq i < j . \tag{6.5}$$

Now we assert

Lemma 7. *Under the restrictions*

$$1 \leqq i < j \leqq \log^{10} n, \quad n > c , \tag{6.6}$$

the relations

$$\begin{aligned} p(n, i, j) &= p(n) - p(n-i) - p(n-j) + p(n-i-j) = \\ &= O\left(\frac{\log^{20} n}{n} p(n)\right) = O\left(\frac{\log^{20} n}{\sqrt{n}} p_1(n)\right) \end{aligned} \tag{6.7}$$

hold.

Proof. The relation

$$1 + \sum_{n=1}^{\infty} p(n, i, j) y^n = \prod_{\substack{\nu=1 \\ \nu \neq i, j}}^{\infty}{}' \frac{1}{1-y^\nu}$$

holds obviously for $0 < y < 1$. From this we get

$$1 + \sum_{n=1}^{\infty} p(n, i, j) y^n = (1-y^i)(1-y^j)\left(1 + \sum_{n=1}^{\infty} p(n) y^n\right),$$

i.e.,

$$p(n, i, j) = p(n) - p(n-i) - p(n-j) + p(n-i-j) \tag{6.8}$$

for $n > i + j$.

Using (6.8), (6.6) and (6.2) we get

$$p(n, i, j) = p(n)\left\{1 - \frac{p(n-i)}{p(n)} - \frac{p(n-j)}{p(n)}\left(1 - \frac{p(n-i-j)}{p(n-j)}\right)\right\} =$$

$$= p(n)\left\{1 - \left(1 + O\left(\frac{i}{n}\right)\right)\exp\left(-\frac{2\pi}{\sqrt{6}} \frac{i}{\sqrt{n} + \sqrt{n-i}}\right) - \right.$$

$$- \left(1 + O\left(\frac{j}{n}\right)\right)\exp\left(-\frac{2\pi}{\sqrt{6}} \frac{j}{\sqrt{n} + \sqrt{n-j}}\right) \times$$

$$\times\left(1-\left(1+O\left(\frac{i}{n}\right)\right)\exp\left(-\frac{2\pi}{\sqrt{6}}\frac{i}{\sqrt{n-j}+\sqrt{n-i-j}}\right)\right)\Bigg\}=$$

$$=p(n)\left\{1-\left(1-\frac{\pi}{\sqrt{6n}}i+O\left(\frac{\log^{20}n}{n}\right)\right)-\right.$$

$$\left.-\left(1-\frac{\pi}{\sqrt{6n}}j+O\left(\frac{\log^{20}n}{n}\right)\right)\left(\frac{\pi}{\sqrt{6n}}i+O\left(\frac{\log^{20}n}{n}\right)\right)\right\}=$$

$$=p(n)O\left(\frac{\log^{20}n}{n}\right)=O\left(\frac{\log^{20}n}{\sqrt{n}}p_1(n)\right).$$

7. Now we turn to the proof of Theorem I. Owing to Lemma 6, we have obviously

$$M(n)\geqq p_1(n)=p(n)-p(n-1)=\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\frac{\pi}{\sqrt{6n}}p(n) \tag{7.1}$$

for $n>c$ and we have to prove only the estimation

$$M(n)-p_1(n)\leqq c\frac{\log^{30}n}{\sqrt{n}}p_1(n) \tag{7.2}$$

for $n>c$, i.e., we have to prove that the $p(n-1)$ partitions of n $(>c)$ containing 1 as summand represent all integers k of the interval $[1, n]$ by subsums apart from

$$c\frac{\log^{30}n}{\sqrt{n}}p_1(n) \tag{7.3}$$

partitions in question at most.

The partitions of n containing 1 as summand represent 1 and we investigate the representations of 2, 3 and 4 for $n>c$.

The number of partitions of n $(>c)$ containing 1 as summand but not representing 2 is obviously

$$=p(n-1, 1, 2)\leqq c\frac{\log^{20}n}{\sqrt{n}}p_1(n) \tag{7.4}$$

owing to Lemma 7.

The number of partitions of n $(>c)$ containing 1 as summand but not representing 3 resp. 4 is obviously

$$\leqq p(n, 2, 3)\leqq c\frac{\log^{20}n}{\sqrt{n}}p_1(n) \tag{7.5}$$

resp.

(7.6)
$$\leqq p(n, 3, 4) \leqq c \frac{\log^{20} n}{\sqrt{n}} p_1(n)$$

owing to Lemma 7.

(7.4), (7.5) and (7.6) yield that the partitions of n ($>c$) containing 1 as summand represent 1, 2, 3 and 4 by subsums apart from

(7.7)
$$c \frac{\log^{20} n}{\sqrt{n}} p_1(n)$$

partitions in question at most.

Next let

(7.8)
$$n > c, \quad 5 \leqq k \leqq \log^{10} n .$$

Taking into consideration Lemma 7 and (7.7) too,

$$k = (k-1)+1 \quad \text{or} \quad k = (k-2)+2 \quad \text{or} \quad k = (k-2)+1+1$$

is a representation of k by a subsum apart from

(7.9)
$$c \frac{\log^{20} n}{\sqrt{n}} p_1(n) + p(n, k-2, k-1) \leqq c \frac{\log^{20} n}{\sqrt{n}} p_1(n)$$

partitions in question at most.

These estimations show that the partitions of n ($>c$) *containing* 1 *as summand* represent *all* integers k of $[1, \log^{10} n]$ by subsums apart from

(7.10)
$$(\log^{10} n)\, c \frac{\log^{20} n}{\sqrt{n}} p_1(n)$$

partitions in question at most. Increasing the constant c we can apply also Lemma 5 for the remaining partitions owing to

$$c p(n)\, n^{-5/4} \log n < c \frac{\log^{30} n}{\sqrt{n}} p_1(n) .$$

After dropping

$$c \frac{\log^{30} n}{\sqrt{n}} p_1(n)$$

exceptional partitions in question at most let Π be an arbitrary partition of n ($>c$) from the remaining ones and k an integer with

(7.11)
$$1 \leqq k \leqq n .$$

We prove by induction that k is representable by a subsum of Π. This assertion has been proved for $1 \leqq k \leqq \log^{10} n$ (and is trivial for $k=n$). We assume that

$$\log^{10} n < k < n \tag{7.12}$$

and that

(7.13) $\begin{cases} \text{all the positive integers less than } k \text{ are representable} \\ \text{by subsums of } \Pi. \end{cases}$

Let

$$\lambda_0 \overset{\text{def}}{=} n \tag{7.14}$$

and define the index $\mu \geqq 0$ by

$$\lambda_\mu > k \geqq \lambda_{\mu-1} \tag{7.15}$$

which makes sense owing to $\lambda_0 > k$ and $1 = \lambda_m < k$. Now, $\lambda_\mu > k > \log^{10} n$ and Lemma 5 preclude the possibility of

$$(n>)k \geqq \lambda_{\mu+1} + \lambda_{\mu+2} + \ldots - \lambda_m \tag{7.16}$$

because (7.16) would imply $\mu \neq 0$ and

$$\lambda_\mu > \lambda_{\mu+1} + \lambda_{\mu+2} + \ldots + \lambda_m$$

in contradiction with (5.13). Therefore, we can define an index v by

$$\lambda_{\mu+1} + \lambda_{\mu+2} + \ldots + \lambda_v + \lambda_{v+1} > k \geqq \lambda_{\mu+1} - \lambda_{\mu+2} + \ldots + \lambda_v . \tag{7.17}$$

This gives that

$$O \leqq k - \lambda_{\mu+1} - \lambda_{\mu+2} - \ldots - \lambda_v < \lambda_{v+1} \leqq \lambda_{\mu+1} \leqq k . \tag{7.18}$$

If $k \neq \lambda_{\mu-1} + \lambda_{\mu+2} + \ldots + \lambda_v$ then (7.13) and (7.18) make it sure that

$$k - \lambda_{\mu+1} - \ldots - \lambda_v = \sum_{j=1}^{s} \lambda_{i_j} < \lambda_{v+1}$$

where

$$v+1 < i_1 < i_2 < \ldots < i_s . \tag{7.19}$$

Then,

$$k = \lambda_{\mu+1} + \ldots + \lambda_v + \lambda_{i_1} + \ldots - \lambda_{i_s}$$

is a representation of k by a subsum owing to (7.19) and Theorem I is completely proved.

8. For the proof of Theorem II we shall use, for

$$\operatorname{Re} z > 0, \tag{8.1}$$

the function

$$f(z) \overset{\text{def}}{=} \prod_{\nu=1}^{\infty} \frac{1}{1-\exp(-\nu z)} = 1 + \sum_{n=1}^{\infty} p(n)\exp(-nz) \tag{8.2}$$

and the well-known formula

$$f(z) = \exp\left(\frac{\pi^2}{6z} + \frac{1}{2}\log\frac{z}{2\pi} + o(1)\right) \quad \text{for} \quad z \to 0 \tag{8.3}$$

in all angles

$$|\arc z| \leqq \kappa < \frac{\pi}{2} \tag{8.4}$$

(log means the principal logarithm).

These give that

$$\prod_{\nu=1}^{\infty}(1+\exp(-\nu z)) = \frac{f(z)}{f(2z)} = \exp\left(\frac{\pi^2}{12z} - \frac{1}{2}\log 2 + o(1)\right) \quad \text{for} \quad z \to 0 \tag{8.5}$$

under the restriction (8.4).

9. We use the notation

$$\Pi^* = \left\{ \begin{matrix} \alpha_1 + \alpha_2 + \ldots + \alpha_m = n \\ \alpha_1 > \alpha_2 > \ldots > \alpha_m \geqq 1 \end{matrix} \right\} \tag{9.1}$$

for a generic *unequal partition* Π^* of n where

$$m = m(\Pi^*) \quad \text{and the } \alpha_\mu\text{'s are integers.} \tag{9.2}$$

According to HARDY–RAMANUJAN's formula (see [1]), the relation

$$q(n) = \frac{1+o(1)}{4n^{3/4}\,3^{1/4}} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right) \tag{9.3}$$

holds for the number $q(n)$ of *unequal partitions* of n.

Then, as it was indicated in **3**, we assert

Theorem II. *Let k_0 be an integer with*

$$1 \leqq k_0 \leqq \frac{n}{2}. \tag{9.4}$$

Then, the unequal partitions of n represent all integers k of the interval $[k_0, n-k_0]$ as subsums apart from

(9.5) $$(20(2/\sqrt{3})^{-k_0}+cn^{-1/10})q(n)$$

unequal partitions of n at most.

10. The proof of Theorem II requires some lemmata.

Lemma 8. *Let k_0 be an integer with*

(10.1) $$1 \leqq k_0 \leqq n^{1/5}$$

Then, for $n>c$, the unequal partitions of n represent all integers k of the interval $[k_0, n^{1/5}]$ as subsums of two terms apart from

(10.2) $$20(2/\sqrt{3})^{-k_0}q(n)$$

unequal partitions of n at most.

Proof. For arbitrary positive integers n and k with $k \geqq 3$, let $q(n,k)$ denote the number of *unequal partitions* of n containing only one of s and $k-s$ *at most* (as summand) for every integer s of the interval $[1,t]$ where $t=[(k-1)/2]$ (i.e., *not* having subsums of the form $(k-s)+s$ with $k-s>s \geqq 1$). We are going to prove that the inequality

(10.3) $$q(n,k)<2(2/\sqrt{3})^{-k}q(n)$$

holds for

(10.4) $$n>c, \quad 3 \leqq k \leqq n^{1/5}$$

Let us observe that the relation

(10.5) $$1+\sum_{n=1}^{\infty} q(n,k)\,w^n=\left\{\prod_{s=1}^{t}\frac{1+w^s+w^{k-s}}{(1+w^s)(1+w^{k-s})}\right\}\prod_{v=1}^{\infty}(1+w^v)$$

holds for $|w|<1$. Cauchy's formula gives the representation

(10.6) $$q(n,k)=\frac{1}{2\pi i}\int\limits_{|w|=\rho} w^{-n-1}\left\{\prod_{s=1}^{t}\left(1-\frac{w^k}{(1+w^s)(1+w^{k-s})}\right)\right\}\prod_{v=1}^{\infty}(1+w^v)\,dw$$

for $0<\rho<1$. Let us define $g_k(z)$ by

(10.7) $$g_k(z)=\left\{\prod_{s=1}^{t}\left(1-\frac{\exp(-kz)}{(1+\exp(-sz))(1+\exp(-(k-s)z))}\right)\right\}\prod_{v=1}^{\infty}(1+\exp(-vz))$$

for

$$x = \operatorname{Re} z > 0 . \tag{10.8}$$

Then we have

$$q(n, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k(x+iy) \exp(n(x+iy))\, dy \tag{10.9}$$

for $x > 0$.

Let C_0 be a sufficiently large constant and ε fixed with $0 < \varepsilon < 10^{-2}$. We choose

$$x = x_0 = \frac{\pi}{2\sqrt{3}} n^{-1/2}, \quad y_1 = n^{-3/4+\varepsilon/3}, \quad y_2 = C_0 x_0 \tag{10.10}$$

and investigate (10.9) as

$$q(n, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k(x_0+iy) \exp(nx_0 + iny)\, dy = \frac{1}{2\pi} \left\{ \int_{-\pi}^{-y_2} + \int_{-y_2}^{-y_1} + \int_{-y_1}^{y_1} + \int_{y_1}^{y_2} + \int_{y_2}^{\pi} \right\}.$$

(We use some ideas of G. A. Freiman's $p(n)$-estimation.)

For

$$n > c, \quad 3 \leqq k \leqq n^{1/5} \quad \text{and} \quad |y| \leqq y_2 , \tag{10.11}$$

we can apply (8.4)–(8.5) and get

$$\prod_{\nu=1}^{\infty} (1 + \exp(-\nu(x_0+iy))) = \exp\left(\frac{\pi^2}{12(x_0+iy)} - \frac{1}{2}\log 2 + o(1) \right) \quad (\text{for } n \to \infty),$$

further,

$$\exp\left(\sum_{s=1}^{t} \log\left(1 - \frac{\exp(-k(x_0+iy))}{(1+\exp(-s(x_0+iy)))(1+\exp(-(k-s)(x_0+iy)))} \right) \right) =$$

$$= \exp\left(\sum_{s=1}^{t} \log\left(1 - \frac{1}{4} + O(kn^{-1/2}) \right) \right) =$$

$$= \exp\left(t \log\frac{3}{4} + O(tkn^{-1/2}) \right) = \exp\left(t \log\frac{3}{4} + O(n^{-1/10}) \right)$$

under the restriction (10.11). Consequently, the relation

$$g_k(x_0+iy) = \exp\left(t\log\frac{3}{4} + \frac{\pi^2}{12(x_0+iy)} - \frac{1}{2}\log 2 + o(1) \right) \tag{10.12}$$

holds under the restriction (10.11).

First,

$$\frac{1}{2\pi}\int_{-y_1}^{y_1} g_t(x_0+iy)\exp(nx_0+iny)\,dy=$$

$$=\frac{1}{2\pi}\int_{-y_1}^{y_1}\exp\left\{t\log\frac{3}{4}+\frac{\pi^2}{12x_0}\left(1-i\frac{y}{x_0}-\left(\frac{y}{x_0}\right)^2+O\left(\left(\frac{y_1}{x_0}\right)^3\right)\right)+\right.$$

$$\left.+nx_0+iny-\frac{1}{2}\log 2+o(1)\right\}dy=$$

$$=\frac{1}{2\pi}\int_{-y_1}^{y_1}\exp\left(t\log\frac{3}{4}+\frac{\pi}{\sqrt{3}}\sqrt{n}-\frac{2\sqrt{3}}{\pi}n^{3/2}y^2+O(n^{-1/4+\varepsilon})-\frac{1}{2}\log 2+o(1)\right)dy=$$

$$=\frac{1+o(1)}{2\pi\sqrt{2}}\left(\frac{3}{4}\right)^t\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right)\int_{-y_1}^{y_1}\exp\left(-\frac{2\sqrt{3}}{\pi}n^{3/2}y^2\right)dy=$$

$$=\frac{1+o(1)}{2\pi\sqrt{2}}\left(\frac{3}{4}\right)^t\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right)\sqrt{\frac{\pi}{2}}\,3^{-1/4}n^{-3/4}\left\{\int_{-\infty}^{+\infty}\exp(-u^2)\,du+o(1)\right\}=$$

$$=\frac{1+o(1)}{4n^{3/4}3^{1/4}}\left(\frac{3}{4}\right)^t\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right)=(1+o(1))\left(\frac{3}{4}\right)^t q(n).$$

Next,

$$\frac{1}{2\pi}\left|\int_{-y_2}^{-y_1}+\int_{y_1}^{y_2}\right|\leqq\frac{1}{\pi}\int_{y_1}^{y_2}\exp\left(t\log\frac{3}{4}+\frac{\pi^2x_0}{12(x_0^2+y^2)}+nx_0+O(1)\right)dy\leqq$$

$$\leqq c\left(\frac{3}{4}\right)^t\exp\left(\frac{\pi^2x_0}{12(x_0^2+y_1^2)}+nx_0\right)=$$

$$=c\left(\frac{3}{4}\right)^t\exp\left(\frac{\pi^2}{12x_0}\left(1-\frac{y_1^2}{x_0^2}+O\left(\frac{y_1^4}{x_0^4}\right)\right)+nx_0\right)\leqq$$

$$\leqq c\left(\frac{3}{4}\right)^t\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}-\frac{2\sqrt{3}}{\pi}n^{2\varepsilon/3}\right)=o(1)\left(\frac{3}{4}\right)^t q(n).$$

Finally, we have to estimate the expression

$$\frac{1}{2\pi}\left|\int_{-\pi}^{-y_2}+\int_{y_2}^{\pi}\right|.$$

For

(10.13) $$n>c,\quad 3\leqq k\leqq n^{1/5}\quad\text{and } y_2\leqq|y|\leqq\pi,$$

we get (with $z=x_0+iy$)

$$\exp\left(\operatorname{Re}\sum_{\nu=1}^{\infty}\log(1+\exp(-\nu z))\right)=\exp\left(\operatorname{Re}\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu-1}}{\mu}\exp(-\nu\mu z)\right)=$$

$$=\exp\left(\operatorname{Re}\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu-1}}{\mu(\exp(\mu z)-1)}\right)\leqq\exp\left(\sum_{\mu=1}^{\infty}\frac{1}{\mu}|\exp(\mu z)-1|^{-1}\right)=$$

$$=\exp\left(\sum_{\mu=1}^{\infty}\frac{1}{\mu}\left((\exp(\mu x_0)-1)^2+4\exp(\mu x_0)\sin^2\frac{\mu y}{2}\right)^{-1/2}\right)\leqq$$

$$\leqq\exp\left(\left(2\sin\frac{|y|}{2}\right)^{-1}+\sum_{\mu=2}^{\infty}\frac{1}{\mu^2x_0}\right)\leqq\exp\left(\frac{1}{x_0}\left(\frac{\pi^2}{6}-1+\frac{\pi}{2C_0}\right)\right)$$

and

$$\prod_{s=1}^{t}\left|1-\frac{\exp(-kz)}{(1+\exp(-sz))(1+\exp(-(k-s)z))}\right|\leqq$$

$$\leqq\prod_{s=1}^{t}\left(1+\frac{1}{(1-\exp(-sx_0))(1-\exp(-(k-s)x_0))}\right)\leqq\left(1+\frac{1}{(1-\exp(-x_0))^2}\right)^t\leqq$$

$$\leqq(cn)^t\leqq\exp(k\log(cn))\leqq\exp(n^{1/4})=\exp\left(\frac{o(1)}{x_0}\right).$$

Therefore,

$$\frac{1}{2\pi}\left|\int_{-\pi}^{-y_2}+\int_{y_2}^{\pi}\right|\leqq\frac{1}{\pi}\int_{y_2}^{\pi}\exp\left(\frac{1}{x_0}\left(\frac{\pi^2}{6}-1+\frac{\pi}{2C_0}+o(1)\right)+nx_0\right)dy\leqq$$

$$\leqq\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}-\frac{2\sqrt{3}}{\pi}\sqrt{n}\left(1-\frac{\pi^2}{12}-\frac{\pi}{2C_0}-o(1)\right)\right)=o(1)\left(\frac{3}{4}\right)^t q(n)$$

owing to (10.10) and (10.13).

Thus we have proved the estimation

$$q(n,k)=(1+o(1))\left(\frac{3}{4}\right)^{t}q(n)<2\left(\frac{2}{\sqrt{3}}\right)^{-k}q(n) \tag{10.14}$$

for

$$n>c,\quad 3\leqq k\leqq n^{1/5}. \tag{10.15}$$

The assertion of Lemma 8 is trivial for $k_0=1, 2$ owing to $20(2/\sqrt{3})^{-2}>1$. For $3\leqq k_0\leqq n^{1/5}$, the total number of the exceptional unequal partitions is

$$\leqq\sum_{k=k_0}^{[n^{1/5}]}q(n,k)\leqq 2q(n)\sum_{k=k_0}^{\infty}\left(\frac{2}{\sqrt{3}}\right)^{-k}<20\left(\frac{2}{\sqrt{3}}\right)^{-k_0}q(n)$$

and Lemma 8 is completely proved.

11. Now we assert

Lemma 9. *The unequal partitions of $n(>c)$ represent all integers k of the interval*

$$\left[n^{1/5},\left(1+\frac{1}{5}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n\right] \tag{11.1}$$

as subsums of four terms at most apart from

$$cq(n)n^{-4} \tag{11.2}$$

unequal partitions of n at most.

Proof. For arbitrary positive integers n and k with $k\geqq 10$ and the notations

$$t_1=\left[\frac{k}{4}\right]+1,\quad t_2=\left[\frac{k-1}{2}\right],\quad t=t_2-t_1+1, \tag{11.3}$$

let $q_1(n,k)$ denote the number of unequal partitions of n containing only one of s and $k-s$ *at most* (as summand) for every integer s of the interval $[t_1,t_2]$ (i.e., *not* having subsums of the form $(k-s)+s$ with $k-s>s>k/4$).

Let us observe that the relation

$$\begin{aligned}1+\sum_{n=1}^{\infty}q_1(n,k)\exp(-nx)=&\\=\left\{\prod_{s=t_1}^{t_2}\frac{1+\exp(-sx)+\exp(-(k-s)x)}{(1+\exp(-sx))(1+\exp(-(k-s)x))}\right\}\prod_{v=1}^{\infty}(1+\exp(-vx))&\end{aligned} \tag{11.4}$$

holds for $x>0$. (11.4) yields that

$$q_1(n,k)\exp(-nx)\leqq$$

$$\leqq\left\{\prod_{s=t_1}^{t_2}\left(1-\frac{\exp(-kx)}{(1+\exp(-sx))(1+\exp(-(k-s)x))}\right)\right\}\prod_{\nu=1}^{\infty}(1+\exp(-\nu x))\leqq$$

$$\leqq\left\{\prod_{s=t_1}^{t_2}\left(1-\frac{1}{4}\exp(-kx)\right)\right\}\prod_{\nu=1}^{\infty}(1+\exp(-\nu x)),$$

i.e.,

$$q_1(n,k)\leqq\left\{\prod_{\nu=1}^{\infty}(1+\exp(-\nu x))\right\}\exp\left(nx+t\log\left(1-\frac{1}{4}\exp(-kx)\right)\right)$$

for $x>0$.

Choosing

(11.5) $$x=x_0=\frac{\pi}{2\sqrt{3}}n^{-1/2}$$

and using (8.4)–(8.5), we get

$$q_1(n,k)\leqq\left\{\prod_{\nu=1}^{\infty}(1+\exp(-\nu x_0))\right\}\exp\left(nx_0-\frac{t}{4}\exp(-kx_0)\right)\leqq$$

$$\leqq c\exp\left(\frac{\pi^2}{12x_0}+nx_0-\frac{k}{16}\exp(-kx_0)\right)=$$

$$=c\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}-\frac{k}{16}\exp(-kx_0)\right)$$

i.e.,

(11.6) $$q_1(n,k)\leqq cq(n)\exp\left(\frac{3}{4}\log n-\frac{k}{16}\exp\left(-\frac{\pi k}{2\sqrt{3n}}\right)\right).$$

For

(11.7) $$n>c,\quad n^{1/5}\leqq k\leqq\frac{2\sqrt{3}}{\pi}\sqrt{n},$$

we get

(11.8) $$q_1(n,k)\leqq cq(n)\exp\left(\frac{3}{4}\log n-\frac{1}{16e}n^{1/5}\right)<cq(n)n^{-5}$$

For

(11.9) $$n>c,\quad \frac{2\sqrt{3}}{\pi}\sqrt{n}\leqq k\leqq\left(1-\frac{1}{100}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n,$$

(11.6) gives the estimation

$$q_1(n, k) \leqq cq(n) \exp\left(\frac{3}{4}\log n - \frac{2\sqrt{3n}}{16\pi}\exp\left(-\left(\frac{1}{2}-\frac{1}{200}\right)\log n\right)\right) =$$

(11.10)

$$= cq(n)\exp\left(\frac{3}{4}\log n - \frac{2\sqrt{3}}{16\pi}n^{1/200}\right) < cq(n)n^{-5}.$$

(11.7)–(11.10) show that the unequal partitions of $n(>c)$ represent all integers k of the interval

(11.11)
$$I_1 = \left[n^{1/5}, \left(1-\frac{1}{100}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n\right]$$

as subsums of the form

(11.12)
$$k = (k-s) + s \quad \text{with} \quad k-s > s > k/4$$

apart from

$$\sum_{\substack{k \in I_1 \\ k \text{ integer}}} q_1(n, k) < cq(n)n^{-4}$$

unequal partitions of n at most. After dropping these exceptional unequal partitions let

$$\Pi^* = \begin{Bmatrix} \alpha_1 + \alpha_2 + \ldots + \alpha_m = n \\ \alpha_1 > \alpha_2 > \ldots > \alpha_m \geqq 1 \end{Bmatrix}$$

be an arbitrary unequal partition of $n(>c)$ from the remaining ones and k an arbitrary integer with

(11.13)
$$\left(1-\frac{1}{100}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n \leqq k \leqq \left(1+\frac{1}{5}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n.$$

Let

$$k_1 = \left[\left(1-\frac{1}{100}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n - n^{1/5}\right], \quad k_2 = k - k_1.$$

Then we can use the property (11.11)–(11.12) of Π^* owing to $k_1, k_2 \in I_1$. Thus,

$$k_1 = \alpha_{u_1} + \alpha_{v_1}, \quad \alpha_{u_1} > \alpha_{v_1} > \frac{k_1}{4},$$

$$k_2 = \alpha_{u_2} + \alpha_{v_2}, \quad \alpha_{u_2} > \alpha_{v_2} > \frac{k_2}{4}.$$

and

$$k_2 < \left(\frac{1}{5} + \frac{2}{100}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n < \frac{k_1}{4}.$$

Therefore,

$$k = \alpha_{u_1} + \alpha_{v_1} + \alpha_{u_2} + \alpha_{v_2} \tag{11.14}$$

is a representation of k by a subsum of Π^* owing to

$$\alpha_{u_1} > \alpha_{v_1} > \frac{k_1}{4} > k_2 > \alpha_{u_2} > \alpha_{v_2}.$$

(11.11)–(11.14) prove Lemma 9.

In order to show that the upper value in (11.1) "usually" exceeds the maximal summand, using the notation (9.1) we assert

Lemma 10. *If $\beta = \beta(n)$ is restricted by*

$$0 < \beta < \frac{\pi}{4\sqrt{3}} \cdot \frac{\sqrt{n}}{\log n} - \frac{1}{2} \tag{11.15}$$

then the inequality

$$\alpha_1 \leqq (1 + 2\beta)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n \tag{11.16}$$

holds with the exception of $cq(n)n^{-\beta}$ Π^*'s at most. In particular, the inequality

$$\alpha_1 \leqq \left(1 + \frac{1}{5}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n \tag{11.17}$$

holds for all but

$$cq(n)n^{-1/10} \tag{11.18}$$

Π^*'s at most.

Proof. In order to estimate the number of the exceptional Π^*'s, let

$$F \overset{\text{def}}{=} \left[(1 + 2\beta)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n\right] + 1.$$

The number of Π^*'s with $\alpha_1 = j$, $F \leqq j \leqq n-1$ is $\leqq q(n-j)$. Hence, the number of the exceptional Π^*'s is

$$\leqq \sum_{j=F}^{n-1} q(n-j) + 1 = 1 + \sum_{l=1}^{n-F} q(l).$$

Using (9.3) we get

$$1+\sum_{l=1}^{n-F} q(l) \leqq c+c \sum_{l=1}^{n-F} l^{-3/4} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{l}\right) \leqq$$

$$\leqq c+c \int_1^{n-F+1} x^{-3/4} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{x}\right) dx \leqq c+c(n-F+1)^{-1/4} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n-F+1}\right).$$

Owing to (11.15), we have

$$n-F+1 \geqq \frac{n}{2}.$$

Consequently,

$$c+c(n-F+1)^{-1/4} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n-F+1}\right) \leqq cq(n)n^{1/2} \exp\left\{-\frac{\pi}{\sqrt{3}}(\sqrt{n}-\sqrt{n-F+1})\right\} =$$

$$= cq(n)n^{1/2} \exp\left\{-\frac{\pi}{\sqrt{3}} \cdot \frac{\left[(1+2\beta)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n\right]}{\sqrt{n}+\sqrt{n-F+1}}\right\} \leqq cq(n)n^{-\beta}.$$

Q.e.d.

12. Continuing the representation by induction, we can see that Lemma 9 and (11.17)–(11.18) preclude the possibility of an inequality analogous to (7.16) (since now $\mu+1$ would be 1). Another difficulty is, however, caused by the lack of the "small" integers representable. In order to avoid this difficulty we assert

Lemma 11. *Dropping*

(12.1) $$cq(n)\exp(-10^{-3}n^{1/12})$$

exceptional unequal partitions of n at most, each Π^* of $n(>c)$ from the remaining ones has a summand in the interval

(12.2) $$\left(\frac{\tau}{2}, \frac{2\tau}{3}\right]$$

for every integer τ restricted by

(12.3) $$10^{-2}\sqrt{n} \leqq \tau \leqq \left(1+\frac{1}{4}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n.$$

Proof. For arbitrary positive integers n and τ with $\tau \geqq 6$, let $q_2(n, \tau)$ denote the number of unequal partitions of n *not* having summands from the interval $(\tau/2, 2\tau/3]$.

Then, the relation

$$1+\sum_{n=1}^{\infty} q_2(n,\tau)\exp(-nx)=$$

(12.4)

$$=\left\{\prod_{\nu=1}^{[\tau/2]}(1+\exp(-\nu x))\right\}\prod_{\mu=[2\tau/3]+1}^{\infty}(1+\exp(-\mu x))$$

holds for $x>0$. (12.4) yields that

$$q_2(n,\tau)\exp(-nx)\leqq\left\{\prod_{\nu=1}^{\infty}(1+\exp(-\nu x))\right\}\prod_{\mu=[\tau/2]+1}^{[2\tau/3]}(1+\exp(-\mu x))^{-1}\leqq$$

$$\leqq\left\{\prod_{\nu=1}^{\infty}(1+\exp(-\nu x))\right\}\left(1+\exp\left(-\frac{2\tau}{3}x\right)\right)^{-[2\tau/3]+[\tau/2]}$$

i.e.,

$$q_2(n,\tau)\leqq\left\{\prod_{\nu=1}^{\infty}(1+\exp(-\nu x))\right\}\exp\left(nx-\left(\left[\frac{2\tau}{3}\right]-\left[\frac{\tau}{2}\right]\right)\log\left(1+\exp\left(-\frac{2\tau}{3}x\right)\right)\right)$$

for $x>0$.

Choosing

$$x=x_0=\frac{\pi}{2\sqrt{3}}n^{-1/2} \tag{12.5}$$

and using (8.4)–(8.5), we get

$$q_2(n,\tau)\leqq c\exp\left(\frac{\pi^2}{12x_0}+nx_0-\frac{\tau}{6}\log\left(1+\exp\left(-\frac{2\tau}{3}x_0\right)\right)\right). \tag{12.6}$$

Taking into consideration (12.3), the estimation (12.6) gives that, for $n>c$,

$$q_2(n,\tau)\leqq c\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}-\frac{1}{600}\sqrt{n}\log\left(1+\exp\left(-\frac{5}{12}\log n\right)\right)\right)\leqq$$

$$\leqq c\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}-\frac{1}{700}n^{1/12}\right)\leqq cq(n)\exp\left(-\frac{1}{800}n^{1/12}\right).$$

This yields that the number of the exceptional Π^*'s is

$$\leqq\sum_{\substack{10^{-2}\sqrt{n}\leqq\tau\leqq\left(1+\frac{1}{4}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n\\ \tau\text{ integer}}} q_2(n,\tau)<cq(n)\exp(-10^{-3}n^{1/12}).$$

Q.e

13. Finally, we assert

Lemma 12. *The unequal partitions of $n(>c)$ represent all integers k of the interval*

$$\left[n^{1/5}, \frac{1}{2}n\right] \tag{13.1}$$

as subsums apart from

$$cq(n)\, n^{-1/10} \tag{13.2}$$

unequal partitions of n at most.

Proof. After dropping

$$cq(n)\, n^{-1/10}$$

exceptional unequal partitions of $n(>c)$ at most Lemma 9, Lemma 11 and (11.17) from Lemma 10 will be applicable. Let

$$\Pi^* = \begin{Bmatrix} \alpha_1 + \alpha_2 + \ldots + \alpha_m = n \\ \alpha_1 > \alpha_2 > \ldots > \alpha_m \geqq 1 \end{Bmatrix}$$

be an arbitrary unequal partition of n from the remaining ones and k an integer with

$$n^{1/5} \leqq k \leqq \frac{1}{2}n. \tag{13.3}$$

We prove by induction that k is representable by a subsum of Π^*. This assertion has been proved for

$$n^{1/5} \leqq k \leqq \left(1 + \frac{1}{5}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n$$

by Lemma 9. We assume that

$$\left(1 + \frac{1}{5}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n < k \leqq \frac{1}{2}n \tag{13.4}$$

and that

$$\begin{cases} \text{all integers of } [n^{1/5}, k-1] \text{ are representable} \\ \text{by subsums of } \Pi^*. \end{cases} \tag{13.5}$$

Owing to (11.17) and (13.4), we have

$$\alpha_1 < k \leqq \frac{1}{2}n < \alpha_1 + \alpha_2 + \ldots + \alpha_m = n. \tag{13.6}$$

Therefore, we can define an index ν by

$$\alpha_1+\alpha_2+\ldots+\alpha_\nu \leqq k < \alpha_1+\alpha_2+\ldots+\alpha_\nu+\alpha_{\nu+1}. \tag{13.7}$$

This gives that

$$0 \leqq \Delta \overset{\text{def}}{=} k-\alpha_1-\alpha_2-\ldots-\alpha_\nu < \alpha_{\nu+1} < \alpha_1 < k. \tag{13.8}$$

For

$$n^{1/5} \leqq \Delta(<k),$$

(13.5) and (13.8) make it sure that

$$k-\alpha_1-\ldots-\alpha_\nu = \sum_{j=1}^{s} \alpha_{i_j} < \alpha_{\nu+1}$$

where

$$\nu+1 < i_1 < i_2 < \ldots < i_s. \tag{13.9}$$

Then,

$$k = \alpha_1+\alpha_2+\ldots+\alpha_\nu+\alpha_{i_1}+\ldots+\alpha_{i_s}$$

is a representation of k by a subsum owing to (13.9). The case $\Delta=0$ is trivial. The only problematic case we have to investigate is

$$1 \leqq \Delta < n^{1/5}. \tag{13.10}$$

We have obviously

$$m < \sqrt{2n} \tag{13.11}$$

$\left(\text{from } n=\alpha_1+\ldots+\alpha_m \geqq m+\ldots+1 > \frac{1}{2}m^2\right)$. (13.4) and (13.7) give that

$$\alpha_{\nu+1}+\ldots+\alpha_m = n-(\alpha_1+\ldots+\alpha_\nu) \geqq n-k \geqq \frac{n}{2},$$

consequently,

$$\frac{n}{2} \leqq \alpha_{\nu+1}+\ldots+\alpha_m \leqq (m-\nu)\alpha_{\nu+1} < \sqrt{2n}\,\alpha_{\nu+1}$$

owing to (13.11). Thus we have

$$\alpha_\nu > \alpha_{\nu+1} > \frac{1}{2\sqrt{2}}\sqrt{n} > 10^{-2}\sqrt{n}. \tag{13.12}$$

Choosing

$$\tau = \alpha_\nu + \Delta \tag{13.13}$$

we get

$$10^{-2}\sqrt{n}<\alpha_\nu<\tau<\alpha_1+n^{1/5}<\left(1+\frac{1}{4}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n \tag{13.14}$$

for $n>c$ from (13.12), (13.10) and (11.17). (13.14) shows that (12.3) is satisfied with the choice (13.13). Applying Lemma 11, we get an index μ with

$$\frac{\alpha_\nu}{2}+\frac{\Delta}{2}=\frac{\tau}{2}<\alpha_\mu\leqq\frac{2\tau}{3}=\frac{2\alpha_\nu}{3}+\frac{2\Delta}{3}. \tag{13.15}$$

Then,

$$\alpha_\mu<\frac{2\alpha_\nu}{3}+n^{1/5}=\alpha_\nu-\left(\frac{1}{3}\alpha_\nu-n^{1/5}\right)<\alpha_\nu-\left(\frac{1}{6\sqrt{2}}\sqrt{n}-n^{1/5}\right)<\alpha_\nu$$

for $n>c$ owing to (13.15), (13.10) and (13.12). Consequently,

$$\nu<\mu. \tag{13.16}$$

Let

$$\Delta_1=\tau-\alpha_\mu. \tag{13.17}$$

Then we get

$$k=\alpha_1+\alpha_2+\ldots+\alpha_{\nu-1}+\alpha_\mu+\Delta_1 \tag{13.18}$$

from (13.8), (13.13) and (13.17). Further,

$$\Delta_1=\tau-\alpha_\mu<2\alpha_\mu-\alpha_\mu=\alpha_\mu<\alpha_1\leqq\left(1+\frac{1}{5}\right)\frac{\sqrt{3}}{\pi}\sqrt{n}\log n \tag{13.19}$$

and

$$\Delta_1=\tau-\alpha_\mu\geqq\frac{1}{3}\tau>\frac{1}{300}\sqrt{n}>n^{1/5}$$

for $n>c$ from (13.17), (13.15), (11.17) and (13.14). Now, we can apply Lemma 9 for Δ_1. This yields that

$$\Delta_1=\sum_{j=1}^{u}\alpha_{r_j} \tag{13.20}$$

with

$$\mu<r_1<r_2<\ldots<r_u \tag{13.21}$$

owing to (13.19). Consequently,

$$k=\alpha_1+\alpha_2+\ldots+\alpha_{\nu-1}+\alpha_\mu+\alpha_{r_1}+\ldots+\alpha_{r_u}$$

is a representation of k by a subsum owing to (13.18), (13.20), (13.16) and (13.21). This settles the case (13.10) and Lemma 12 is completely proved.

Now, Lemma 8 and Lemma 12 prove Theorem II for $k_0 \leqq k \leqq \frac{1}{2} n$ and for $\frac{1}{2} n < k \leqq n - k_0$ too by means of the complementary subsums.

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Studies in Pure Mathematics
To the Memory of Paul Turán

On sums and products of integers

by

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Let $1 \leqq a_1 < \ldots < a_n$ be a sequence of integers Consider the integers of the form

$$a_i + a_j, \quad a_i a_j, \quad 1 \leqq i \leqq j \leqq n. \tag{1}$$

It is tempting to conjecture that for every $\varepsilon > 0$ there is an n_0 so that for every $n > n_0$ there are more than $n^{2-\varepsilon}$ distinct integers of the form (1). We are very far from being able to prove this, but we prove the following weaker

Theorem 1. *Denote by $f(n)$ the largest integer so that for every $\{a_1, a_2, \ldots, a_n\}$ there are at least $f(n)$ distinct integers of the form* (1). *Then*

$$n^{1+c_1} < f(n) < n^2 \exp(-c_2 \log n / \log\log n). \tag{2}$$

We expect that the upper bound in (2) may be close to the "truth".

More generally we conjecture that for every k and $n > n_0(k)$ there are more than $n^{k-\varepsilon}$ distinct integers of the form

$$a_{i_1} + \ldots + a_{i_k}, \quad \prod_{j=1}^{k} a_{i_j}$$

At the moment we do not see how to attack this plausible conjecture.

Denote now by $g(n)$ the largest integer so that for every $\{a_1, \ldots, a_n\}$ there are at least $g(n)$ distinct integers of the form

$$\sum_{i=1}^{n} \varepsilon_i a_i, \quad \prod_{i=1}^{n} a_i^{\varepsilon_i} \quad (\varepsilon_i = 0 \text{ or } 1) \tag{3}$$

We conjecture that for $n > n_0(k)$, $g(n) > n^k$. Unfortunately we have not been able to prove this and perhaps we overlook a simple idea. We prove