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POLYCHROMATIC EUCLIDEAN RAMSEY THEORÉMS

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Dedicated to Rafael Artzy on the occasion of his 70th birthday

1. Introduction

In earlier papers [1, 2, 3] we discussed sets S with the Euclidean Ramsey Property, ERP, that is sets S in a Euclidean space E^n with the property that for every r there exists an $N = N(r,s)$ so that in every r-coloring of E^m with $m \geq N$, there exists a monochromatic set S' congruent to S. We proved [1] that a necessary condition for the Euclidean Ramsey Property is that S is a finite subset of a sphere. More generally, if S has a k-chromatic congruent copy in all r-colorings of sufficiently high dimensional Euclidean spaces (we call this property k-ERP) then S must be embeddable in k concentric spheres.

In this note we describe some sets which have this k-ERP but not the $(k-1)$-ERP. In the process it is useful to consider simplicial colorings of a Euclidean space, that is cllorings which simultaneously color each set of congruent ℓ-tuples of E^m in a fixed number of colors.

1.1 Definition. A finite set S in E^n has simplicial ERP if for the colorings of the sets congruent to the subsets of S with a fixed number of colors there exists an N so that for all $m \geq N$ there is a set $S' \subset E^m$ congruent to S so that all its congruent subsets have the same color.

2. Direct consequences of Ramsey's Theorem

2.1 Theorem. The regular simplex S_n has the simplicial ERP.

With the help of Theorem 2.1 we can construct a variety of point sets with the exact k-ERP, that is k-chromatic but not $(k-1)$-chromatic congruent copies exist in every high-dimensional r-colored Euclidean space. For example we have the following.

2.2 Corollary. Let $0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n-1$ and let P_i denote the set of centroids of the i-subsimplices of a regular simplex S_n. Then the set $S = P_{i_1} \cup P_{i_2} \cup \cdots \cup P_{i_k}$ has the exact k-ERP. The coloring can be chosen so that each $P_i \subset S$ has a monochromatic copy.

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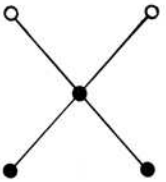
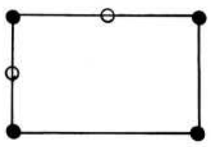
Proof. The fact that S has the k-ERP follows directly from Theorem 2.1 where each S_i is given the color of its centroid. It is easy to see that the concentric spheres which contain P_{i_j} and whose center is the centroid of S_n constitute a minimal set of concentric spheres containing S. Thus S does not have the (k - 1)-ERP.

The examples of Corollary 2.2 by no means exhaust the possibilities of applying Ramsey's Theorem to the Euclidean case. These possibilities exist whenever we have a highly transitive group of isometries (i.e. either the alternating or the symmetric group) acting on a family of subsets of a large set. For example, we can restrict attention to the isometries of a regular n-simplex S_n which keep some sub-simplex S_i fixed and then we can color the sub-simplices $S_j \supset S_i$ by the color of any point in the $(i+1)$-plane through S_i and the centroid of S_j.

2.3 Example. In the case $S_i = S_0$ and $S_j = S_1$ we get a set with the exact 2-ERP by picking the vertices of two regular n-simplices which have one common vertex and whose edges through that vertex are collinear. We can pick the coloring so that the vertices of one simplex are monochromatic and the remaining vertices are monochromatic. The illustration shows the case $n = 2$.

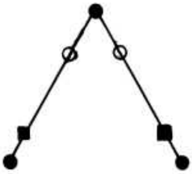


The same reasoning can be applied to a rectangular solid with a large number of congruent sub-boxes through a common sub-box. We illustrate it again by picking edges through a vertex. We then can get a set with the exact 2-ERP by taking the vertices of a cube and, say, the midpoints of the edges through one vertex.



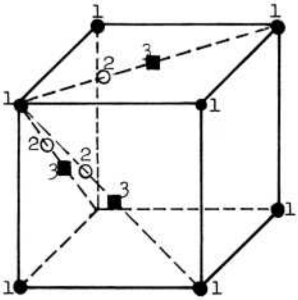
There was no need to color the sub-simplices or sub-boxes in terms of the colors of single points.

2.4 Example. The set of vertices of a regular (unit) n-simplex together with the points that divide the edges incident to one vertex into consecutive segments of length $x_1, x_2, \ldots, x_{k-1}$ starting from the common vertex have the k-ERP.



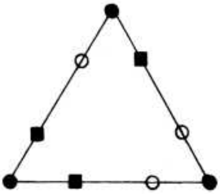
We can only prove that this set does not have the $(k-1)$-ERP when none of the division points are symmetric about the centers of the edges.

2.5 Example. The vertices of a cube together with the points that are one third and one half the length along the diagonal of a 2-face from a fixed vertex have the exact 3-ERP.



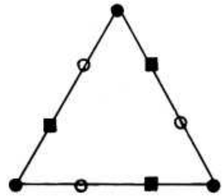
We give one final example, applying Theorem 2.1 where we color the edges of an n-simplex according to the (unordered) pair of colors of its trisecting points.

2.6 Example. The vertices of an n-simplex plus the trisection points of its edges have the 3-ERP. This point set lies in the union of two concentric spheres. It therefore does not have the ERP but we cannot prove that it does not have 2-ERP.



Incidentally, this shows that the vertices of every equiangular hexagon with alternate sides of equal length has the 2-ERP.

We note that there are essentially two different arrangements possible for the colorings in Example 2.6, the one shown above, and the following "cyclic" arrangement:



By considering oriented segments we can guarantee that the first of these occurs. Let all the unit segments be ordered, and color them according to the ordered sequence of endpoints and trisection points. By considering a large simplex (in a high enough dimensional space) we actually have a large tournament with the (directed) edges colored. Then we can find a monochromatic transitive subtournament (by the pigeon hole principle), in this case on three vertices. The resulting configuration is the first of the two arrangements.

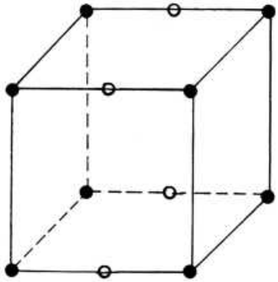
It may be that the second arrangement also has the 3-ERP. This is still an open question. However, the use of directed segments by itself isn't sufficient to settle the question. For this method requires that if the unit segments are all oriented and r-colored, then we want to find a monochromatic cyclically ordered unit triangle. Even if $r = 2$ this can be prevented as follows. Let all the points be ordered. Then orient each unit segment from the lower to the higher vertex. All unit triangles are of the "transitive" or non-cyclic variety. In fact, what this example shows is that if the unit segments are oriented and r-colored, there will always be a monochromatic configuration of type K only if K is acyclic. This still leaves open some of the more geometric questions such as the existence of monochromatic unit rhombuses with edges ordered, say, (a,b), (a,c), (b,d), (c,d), and with fixed angle.

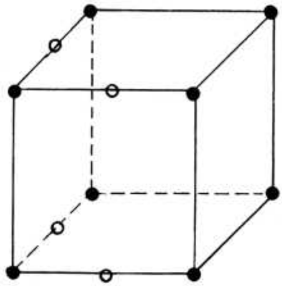
3. Geometric considerations

3.1 Theorem. *If S has ERP and T has k-ERP then $S \times T$ has k-ERP.*

Proof. The proof is completely analogous to [1]. If T has k-ERP then for every r there must exist a finite set U so that in every r-coloring of U there exists a k-chromatic congruent copy of T. We now embed $S \times U$ in a sufficiently high dimensional r-colored Euclidean space and assign to each point of S the coloring of U. This gives an $r^{|U|}$-coloring of S and since S has ERP it follows that in high enough dimensions there exists a copy of S so that all $s \times U$, $s \in S$ are colored the same. Thus we can pick corresponding k-chromatic subsets $s \times Y$ colored with the same k colors for all $s \in S$.

3.2 *Example*. As an immediate consequence of Theorem 3.1 we see that the vertices of a cube together with the midpoints of 4 edges have the 2-ERP in the following cases.





3.3 <u>Definition</u>. An isosceles n-simplex S_n is a simplex containing a regular $(n-1)$-simplex as sub-simplex called the base of S_n and the remaining vertex, called the apex of S_n is equidistant from the vertices of the base.

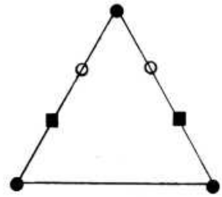
3.4 <u>Theorem</u>. <u>An isosceles n-simplex S_n, whose edges at the apex form non-obtuse angles, has the simplicial</u> ERP.

<u>Proof</u>. We can embed S_n in an isosceles S_N with the same apex and arbitrarily large N. The group of isometries of S_N is the full permutation group on each of the isometry classes of sub-simplices of S_N, except for the vertices where it keeps the apex fixed (unless S_n is regular). It therefore suffices to prove that the set of vertices of S_N has the ERP. According to Theorem 3.1 it suffices to prove that we can embed the vertices of S_N in the vertices of a box (rectangular parallelepiped). For this purpose we choose the origin 0 as a vertex of the box corresponding to the apex of S_N and let a sufficiently large number of edges incident at 0 have lengths a and the remaining ones have lengths b. So the vertices of the box are points $(x_1, x_2, \ldots, x_{N_1}, y_1, y_2, \ldots, y_{N_1})$ where each x_i is 0 or a and each y_j is 0 or b. Now consider the vertices with $x_1 = a$, $x_i = a$, $y_i = b$ and the rest of the coordinates 0. These vertices have distance $\sqrt{2a^2 + b^2}$ from 0 and $\sqrt{2a^2 + 2b^2}$ from each other. Thus, if $N_1 > N$ we can embed any isosceles S_N with acute apex angles. The right isosceles S_N is given by one corner of an N-cube and the adjacent vertices.

One could construct other simplicially ERP sets such as certain pairs of isosceles simplices with a common base.

As in Corollary 2.2, one can color each simplex by the colors of a finite set of points which are invariant under its isometries and in this way construct varieties of point sets which (exactly) have the k-ERP.

3.5 <u>Example</u>. The vertices of non-obtuse isosceles triangle and the trisecting points of its sides have the 3-ERP, but not the 2-ERP if the triangle is not equilateral. The colors can be assigned as in the figure.



4. Some problems and additional results

4.1 Consider four collinear points which lie symmetric about a point. That is, points on the x-axis with coordinates $-b$, $-a$, a, b where $0 < a < b$. Does such a set have the 2-ERP for any (all) values of a, b? We can prevent any 2-coloring except those where the outer and inner pairs are of the same color.

4. In connection with Theorem 3.4 we may consider an obtuse angle α. Then we can ask whether the isosceles triangle with apex angle α has the simplicial ERP. But an even simpler question is as follows: Color the unit segments of E^n with r colors. Then for any r is there an n so that we have a monochromatic pair of unit segments making an angle α? Even for $r = 2$ we can prevent this for $\alpha = 180°$. However for $90° < \alpha < 180°$ a new result of R.L. Graham (to appear) shows that for each r there is an n for which there must always be a monochromatic pair of unit segments forming an angle α.

4.3 It is by no means obvious whether the direct product of two sets S, T with simplicial ERP has simplicial ERP. We cannot even prove that in an r_1-coloring of the points and an r_2-coloring of the unit segments there will exist a unit square in a sufficiently high dimensional Euclidean space with monochromatic vertices and edges. However, if S, T have simplicial ERP then the sets $S_1 \times T_1$, $S_1 \subset S$, $T_1 \subset T$ have the ERP, in the sense that for any r-coloring of the congruence classes of sets of that form in a sufficiently high dimensional Euclidean space there exists a set congruent to $S \times T$ so that, if $S_1, S_1' \subset S$ and $T_1, T_1' \subset T$ with $S_1 \cong S_1'$, $T_1 \cong T_1'$, then $S_1 \times T_1$ and $S_1' \times T_1'$ have the same color.

This shows that for the square we can get a copy with monochromatic vertices and opposite pairs of edges colored alike. Thus the vertices and midpoints of the edges of a square have the 3-ERP.

4.4 Are there sets S which have the ERP but not the simplicial ERP? The necessary condition, that S is embedded in a sphere of some E^n, implies that the congruent k-tuples, considered as points of E^{kn}, are embeddable in a sphere. The sufficient condition, that S is embedded in the vertices of a box of E^n, implies that the k-tuples of S are embedded in the vertices of a box of E^{kn}. However the group of isometries of E^{kn} induced by the isometries of E^n is not the full group of isometries of E^{kn}. Thus this does not constitute a known sufficient condition for congruent k-tuples of S to satisfy the ERP.

4.5 Most of the examples of S with k-ERP really consisted of a decomposition of S into disjoint sets $S_1,\ldots,S_k$ which have simultaneous ERP. That is, in any r-coloring of sufficiently high dimensional Euclidean space there exists an S' congruent to S so that the corresponding subsets $S_1',\ldots,S_k'$ congruent to $S_1,\ldots,S_k$ are monochromatic. The set constructed in Example 2.6 was not constructed in this manner. Nonetheless all sets with k-ERP are obtained as unions of k sets with simultaneous ERP.

4.6 Theorem. *If* S *has* k-*ERP then* S *is the disjoint union of* k *sets with simultaneous* ERP.

Proof. Assume otherwise. Then for each of the subdivisions of S into k disjoint sets $S_1,\ldots,S_k$ there exist r-colorings of all E^N; $N = 1,2,\ldots$; with some fixed r, such that in none of the E^N there is a congruent copy of S in which all the copies of the S_i are monochromatic.

By superimposing the colorings (or the different subdivisions of S) we get an R-coloring of all E^N; $N = 1,2,\ldots$ with $R < r^{k^{|S|}}$ so that no E^N contains a k-colored congruent copy of S.

On p. 554 of the paper [2] on Euclidean Ramsey Theorem there is an inaccuracy. The problem in question states:

Let f(n) be the smallest integer with the property that if one colors the segments of length one of f(n)-dimensional space with two colors, there always is an n-dimensional simplex of size 1 all of whose edges are monochromatic. Clearly

$$f(n) \le r(n) \le \binom{2n-2}{n-1}$$

(where r(n) is the Ramsey number for coloring edges of a complete graph and getting a monochromatic complete n-gon) but it is quite possible that $f(n) = o[r(n)]$ and it is not even clear that f(n) tends to infinity exponentially.

Also it would be interesting to decide whether there is an n so that if we color the edges of length one by two colors there always is a monochromatic unit square or even any three of the four sides of a square.

On the same page, 554, it is also stated that we have not settled the existence of the edges of a unit square in the plane -- in fact it is easy to see that the edges of a unit square (or in fact of any square [if we color the segments of any length]) can be avoided in one color.

References

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4. E.G. Straus, A Combinatorial Theorem in Group Theory, Math. of Comp. 29 (1975), 303-309.

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