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PROBLEMS AND RESULTS ON POLYNOMIALS AND INTERPOLATION

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This is not a survey paper. I lost touch with too many of the recent developments to have the courage to attempt writing a survey paper. I will restrict myself to discuss some of the problems which interested me – some of them for several decades and to state some old an new problems. In the first chapter I will discuss some problems on polynomials and in the second problems on interpolation.

I.

1. An old paper of Herzog, Piranian and myself contains many problems and results on polynomials. Many of these were proved or disproved by Pommerenke; the interested reader should also consult a paper by Elbert. Here I only state one of our favourite problems which has remained almost completely unexplored.

Let $f_n(z) = \prod_{i=1}^{n} (z - z_i)$. Denote by E_f the set of points for which $|f_n(z)| \leqslant 1$. We conjecture that the length of the curve $|f_n(z)| = 1$, (i.e.

the boundary of E_f) is maximal for $f_n(z) = z^n - 1$. I offer 250 dollars for a proof or disproof.

P. Erdős – F. Herzog – G. Piranian, Metric properties of polynomials, I, *Analyse Math.*, 6 (1958), 125-148.

Ch. Pommerenke, On some metric properties of polynomials with real zeros, I and II, *Michigan Math. Journal,* 6 (1959), 377-380 and 8 (1961), 49-54.

Ch. Pommerenke, On some problems by Erdős, Herzog and Piranian, *ibid,* 6 (1959), 221-225.

Ch. Pommerenke, In the derivative of a polynomial, *ibid,* 6 (1959), 373-375.

Ch. Pommerenke, In metric properties of complex polynomials, *ibid,* 8 (1961), 97-115.

Ch. Pommerenke, Einige Sätze über die Kapazität ebener Mengen, *Math. Annalen,* 141 (1960), 143-152.

Á. Elbert, Über eine Vermutung von Erdős betreffs Polynome, I and II, *Studia Sci. Math. Hungar.*, 1 (1966), 119-128 and 3 (1968), 299-324.

G. Piranian, The length of a lemniscate, *Amer. Math. Monthly,* 87 (1980), 555-556.

2. Let $|f_n(\vartheta)| \leqslant 1$ $(0 < \vartheta < 2\pi)$ be a trigonometric polynomial of degree n. I proved that the length of arc of these polynomials is maximal if $f_n(\vartheta) = \cos n\vartheta$. I conjectured that if $|P_n(x)| \leqslant 1$ $(-1 < z < 1)$ is a real polynomial of degree n then its length is maximal for $P_n(z) = T_n(z)$, the n-th Tchebyshev polynomial. This conjecture was recently proved and extended by Kristiansen.

P. Erdős, An extremum problem concerning trigonometric polynomials, *Acta Sci. Math. Szeged,* 9 (1939), 113-115.

For some related problems see

P. Erdős, Note on some elementary properties of polynomials.

Some of the problems stated in this paper were settled and in fact generalized by ,

E.B. Staff – T. Sheil-Small, Coefficient and integral mean estimates for algebraic and trigonometric polynomials with restricted zeros, *J. London Math. Soc.*, 9 (1974), 16-22.

G.K. Kristiansen, Some inequalities for algebraic and trigonometric polynomials, *J. London Math. Soc.*, 20 (1979), 300-314.

B.D. Bojanov, Proof of a conjecture of Erdős about the longest polynomial, *Proc. Amer. Math. Soc.*, 84 (1982), 99-103.

3. Let $|z_n| = 1$ $(n = 1, 2, \ldots)$. Put

$$f_n(z) = \prod_{i=1}^{n} (z - z_i), \quad A_n = \max_{|z|=1} \Big| \prod_{i=1}^{n} (z - z_i) \Big|.$$

I conjectured that $\limsup_{n \to \infty} A_n = \infty$. This problem really belongs to the theory of irregularities of distributions, a theory which was started by Van der Corput and van Aardenne-Ehrenfest and which was brilliantly continued by K.F. Roth and W. Schmidt and others.

My conjecture was recently proved by G. Wagner. He showed (using methods of W. Schmidt) that for infinitely many n, $A_n > (\log n)^c$. Very recently Loxton proved that for infinitely many n, $A_n > \exp\left(\frac{\log n}{\operatorname{loglog} n}\right)$. I conjecture that $A_n > n^c$ for some $c < 1$. In fact perhaps for every n

$$(1) \qquad \sum_{i=1}^{n} A_i > n^{1+c}.$$

V.T. Sós conjectured that $A_n \to \infty$ if we neglect a sequence n_ν, which tends to infinity faster than any power of ν.

Put now $B_k = \sup \Big| \sum_{i=1}^{n} z_i^k \Big|$. B_k can, of course, be infinite for every

k but if $z_k = e^{2\pi i k\alpha}$ and α is a quadratic irrationality then $B_k < c_1 k$ for every k. I conjectured that there is an absolute constant c_2 so that for infinitely many k $B_k > c_2 k$. This attractive conjecture is still open. Clunie and independently Kátai proved that $B_k > ck^{\frac{1}{2}}$. Perhaps

$$(2) \qquad \sum_{k=1}^{n} B_k > cn \log n.$$

$z_k = e^{2\pi i k\alpha}$ shows that this conjecture, if true, is best possible.

V.T. Sós conjectured that there is an absolute constant $c > 0$ so that for every k_0 there is an infinite sequence $n_1 < n_2 < \dots$ for which there is a $k(n_j) > k_0$ so that

$$(3) \qquad \sum_{i=1}^{n_j} z_i^{k(n_j)} > ck(n_j).$$

(3) holds if my conjecture $B(k) > c_2 k$ is true.

V.T. Sós further stated the following stronger conjecture. Denote by $A(n)$ the number of integers k for which

$$(4) \qquad \sum_{i=1}^{n} z_i^k > ck.$$

Then

$$(5) \qquad \sup_n A(n) = \infty.$$

As far as we can tell (5) does not follow from my conjecture and does not imply it.

Let $|z_1| = 1$, $|z_i| \leqslant 1$ $(i = 1, 2, \dots, n)$. Turán conjectured more than 40 years ago that if $s_k = \sum_{i=1}^{n} z_i^k$, then

$$(6) \qquad \max_{1 \leqslant k \leqslant n} |s_k| > c.$$

This was the starting point for Turán's famous power sum method.

(6) was proved by Atkinson with $c = \frac{1}{6}$; he later improved this to $\frac{1}{3}$. Turán conjectured that for every $\epsilon > 0$ there is an n_0 so that for every $n > n_0$

(7) $$\max_{1 \leqslant k \leqslant n} |s_k| > 1 - \epsilon.$$

(7), if true, is of course best possible. It would perhaps be of interest to estimate $\sum_{k=1}^{n} |s_k|$. In fact put

$$f(n) = \min \sum_{k=1}^{n} |s_k|$$

where the minimum is extended over all $|z_1| = 1$, $|z_i| \leqslant 1$, and

$$F(n) = \min \sum_{k=1}^{n} |s_k|$$

where the minimum is extended over all $|z_1| = 1$, $|z_i| \geqslant 1$. Is it true that $f(n) \to \infty$ and perhaps $F(n) > n^{\epsilon}$ or even $F(n) > cn$? As far as I know these questions have not yet been investigated and perhaps they are trivial or false.

G. Wagner, On a problem of Erdős in Diophantine approximation, *Bull. London Math. Soc.*, 35 (1960).

J. Clunie, On a problem of Erdős, *J. London Math. Soc.*, 42 (1967), 133-136.

I. Kátai, An irregularity phenomenon in the theory of numbers, (Hungarian, English summary) *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.*, 17 (1967), 85-88.

F.V. Atkinson, On sums of powers of complex numbers, *Acta Math. Acad. Sci. Hungar.*, 12 (1961), 185-188.

For further details on the power sum method and a large number of problems and applications see the forthcoming book of Turán.

4. D. Newman and I conjectured that there is an absolute constant c so that for every choice of $\epsilon_k = \pm 1$

$$\max_{|z|=1} \left| \sum_{k=1}^{n} \epsilon_k z^k \right| > (1+c) n^{\frac{1}{2}}. \tag{1}$$

(1) is still open. We also asked: Does (1) remain true if the condition $\epsilon_k = \pm 1$ is replaced by $|\epsilon_k| = 1$?

Recently Kahane disproved this. In fact he proved that for every $n > n_0(\epsilon)$ there is a polynomial $\sum_{k=0}^{n} a_k z^k$, $|a_k| = 1$ $(0 \leqslant k \leqslant n)$ for which

$$\max_{|z|=1} \left| \sum_{k=0}^{n} a_k z^k \right| < n^{\frac{1}{2}} + n^{\frac{3}{10}+\epsilon}. \tag{2}$$

In fact he proved that there is such a polynomial for which for every $|z| = 1$

$$n^{\frac{1}{2}} - n^{\frac{3}{10}+\epsilon} < \left| \sum_{k=0}^{n} a_k z^k \right| < n^{\frac{1}{2}} + n^{\frac{3}{10}+\epsilon}. \tag{2'}$$

Does (2) or (2′) remain true if $n^{\frac{3}{10}+\epsilon}$ is replaced by n^{ϵ}, or by $O(1)$? It is perfectly possible that this holds for (2) but not for (2′).

I proved some time ago that (1) is true for trigonometric polynomials. Many unsolved problems remain here too.

J.P. Kahane, Sur les polynomes a coefficients unimodularies, *Bull. London Math. Soc.*, 38 (1980), 321-342.

D. Newman, Norms of polynomials, *Amer. Math. Monthly*, 67 (1960), 778-779.

P. Erdős, An inequality for the maximum of trigonometric polynomials, *Ann. Polonii Math.*, 12 (1962), 151-154.

5. Finally I state some miscellaneous old problems of mine (some of them may be trivial or false or well known).

a. Let $|z_i| \leqslant 1$ $(1 \leqslant i \leqslant n)$. Is there a polynomial $P_{n+1}(x)$ of degree $n+1$ all of whose roots have absolute values not exceeding $3^{\frac{1}{2}}$ and for which $P'_{n+1}(z_i) = 0$ $(1 \leqslant i \leqslant n)$?

b. Let $-1 \leqslant x_1 < \ldots < x_n \leqslant 1$, $\omega_n(x) = \prod_{i=1}^{n} (x - x_i)$. Determine

$$\min_{x_1, \ldots, x_n} \max_{1 \leqslant k \leqslant n} |\omega_n'(x_k)|.$$

c. To end this chapter here is a very nice old problem of Sendov: Let $P(z) = \prod_{i=1}^{p} (z - z_i)$, $|z_i| \leqslant 1$. Is it true that $P'(z)$ always has a zero in $|z - z_i| \leqslant 1$?

A good source for problems on polynomials and function theory is:

W.K. Hayman, *Research problems in function theory*, University of London, Athlone Press, 1967.

II.

Let $-1 \leqslant x_1 < \ldots < x_n \leqslant 1$. Denote by $l_k(x)$ the unique polynomial of degree $n-1$ for which

$$l_k(x_k) = 1, \quad l_k(x_i) = 0 \qquad (i \neq k.$$

The $l_k(x)$ are called the fundamental functions of Lagrange interpolation. Put

$$(1) \qquad L_n(x) = \sum_{k=1}^{n} |l_k(x)|, \qquad \min_{-1 \leqslant x_1 < \ldots < x_n \leqslant 1} \max_{-1 \leqslant x \leqslant 1} L_n(x) = A_n.$$

A classical conjecture of S. Bernstein states that $\max_{-1 \leqslant x \leqslant 1} L_n(x) = A_n$ is assumed if and only if all the $n+1$ maxima of $L_n(x)$ in (x_i, x_{i+1}) $(-1 = x_0,\ 1 = x_{n+1})$ are equal. I conjectured that the smallest of these $n+1$ maxima is maximal if all these $n+1$ maxima are equal. In other words for every $-1 \leqslant x_1 < \ldots < x_n \leqslant 1$ one of these $n+1$ maxima is $\geqslant A_n$ and another is $\leqslant A_n$. Equality if and only if all of them are equal.

These conjectures were recently all settled in two fundamental papers by Kilgore, De Boor and Pinkus. Several interesting unsolved extremal problems remain. Szabados and I proved that for $-1 \leqslant a < b \leqslant 1$

$$\int_a^b L_n(x) > c(b-a) \log n \quad \text{for} \quad n > n_0(a, b). \tag{2}$$

We could not determine the largest c for which (2) holds. We expect that the roots of the Tchebyshev polynomials give (asymptotically) the smallest value of c.

Vértesi and I proved with the help of Halász that there are constants c_1 and c_2 so that the measure of the set in x $(-1 \leqslant x \leqslant 1)$ for which

$$L_n(x) > c_1 \log n \tag{3}$$

is at least c_2. We are very far from being able to determine the exact dependence of c_1 from c_2. Presumably the extremal case is again given by the roots of the Tchebyshev polynomials.

Perhaps the following result holds: Let $\{x_i^{(n)}\}$ be a triangular matrix. Then for every $\epsilon > 0$ and almost all x_0 $(-1 \leqslant x_0 \leqslant 1)$

$$\sum_{k=1}^{n} |l_k^{(n)}(x_0)| > \left(\frac{2}{\pi} - \epsilon\right) \log n \tag{4}$$

holds for infinitely many n. The best we could hope for is that in (4) the right side is replaced by $\frac{2}{\pi} \log n - c$.

Presumably for every $c_1 < \frac{2}{\pi}$ (3) holds for a set of positive measure i.e. for every $c_1 < \frac{2}{\pi} - \epsilon$, $c_2 > \delta$, $\delta = \delta(\epsilon)$. We are very far from being able to prove this.

The sharpest conjecture which has a chance of being true states as follows: Let

$$x_i^{(n)} \quad (1 \leqslant i \leqslant n;\ n = 1, 2, \ldots;\ -1 \leqslant x_1^{(n)} < \ldots < x_n^{(n)} \leqslant 1)$$

be a triangular matrix. Then for every $\epsilon > 0$ and almost all x $(-1 \leqslant x \leqslant 1)$

$$(3) \qquad L_n(x) > \left(\frac{2}{\pi} - \epsilon\right) \log n$$

holds for infinitely many n. Perhaps $\left(\frac{2}{\pi} - \epsilon\right) \log n$ in (3) can be replaced by $\frac{2}{\pi} \log n - c$.

I proved some time ago that if

$$\{x_i^{(n)}\} \quad (1 \leqslant i \leqslant n;\ n = 1, 2, \ldots;\ -1 \leqslant x_1^{(n)} < \ldots < x_n^{(n)} \leqslant 1)$$

is a triangular matrix then for almost all x_0

$$\overline{\lim} \sum_{k=1}^{n} |l_k^{(n)}(x_0)| = \infty.$$

More precisely I showed that for every A and ϵ if $n > n_0(\epsilon, A)$ then the measure of the set in x for which $(-1 \leqslant x_1 < \ldots < x_n \leqslant 1)$

$$(4) \qquad \sum_{k=1}^{n} |l_k(x)| < A$$

is less than ϵ.

It is well known that the unboundedness of $\sum_{k=1}^{n} |l_k(x_0)|$ is the necessary and sufficient condition that there is a continuous function $f(x)$ so that the sequence of Lagrange interpolation polynomials $L_n(f(x))$ taken at $(x_i^{(n)})$ should diverge at x_0. Vértesi and I recently proved that for every triangular matrix $\{x_i^{(n)}\}$ there is a continuous $f(x)$ so that for almost all x $L_n(f(x))$ diverges. Vértesi extended this for complex and trigonometric interpolation. The proofs are long and complicated and will appear in *Acta Math. Acad. Sci. Hungar.*

The following interesting but somewhat technical problem remains. Is there a point group $\{x_i^{(n)}\}$ so that for every continuous function $f(x)$ the sequence $L_n(f(x))$ should converge to $f(x)$ for some x_0 for which

$$(5) \qquad \overline{\lim} \sum_{k=1}^{n} |l_k^{(n)}(x_0)| = \infty?$$

I claimed that I can do this but we unsuccessfully tried with Vértesi to reconstruct my "proof" – perhaps the proof was incorrect or incomplete. The following further problem seems of interest. Does there exist a point group for which (5) holds for all x_0 but for every continuous $f(x)$ $L_n(f(x))$ converges to $f(x)$ for at least one value of x? A classical result of G. Grünwald and I. Marcinkiewicz states that the roots of the Tchebyshev polynomials do not have this property.

Before ending the paper I would like to call attention to a mistake in one of my papers with G. Grünwald which I hope to correct before "I leave". Let $\{x_i^{(n)}\}$ be the roots of the Tchebyshev polynomials, $f(x)$ $(-1 \leqslant x \leqslant 1)$ a continuous function and $L_n(f(x))$ the Lagrange interpolation polynomials taken at the roots of $T_n(x)$. We "proved" that there is a continuous function $f(x)$ for which for all x the sequence

$$(6) \qquad \frac{1}{n} \sum_{k=1}^{n} L_k(f(x))$$

diverges and in fact the sequence (6) is unbounded. Later I noticed that we only proved that

$$(6') \qquad \overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{k=1}^{n} L_k(f(x))| = \infty$$

holds for all x. I expect that (6′) holds if $|L_k(f(x))|$ is replaced by $L_k(f(x))$ but there seem to be serious technical difficulties.

P. Erdős, Problems and results on the convergence and divergence properties of the Lagrange interpolation polynomials and some extremal problems, *Mathematica Cluj*, 10 (33) (1968), 65-73.

P. Erdős, On the boundedness and unboundedness of polynomials, *Journal d'Analyse*, 19 (1967), 135-148.

T.A. Kilgore, A characterization of the Lagrange interpolating projection with minimal Tchebyshev norm, *J. Approximation Theory*, 24 (1978), 273-288.

C. De Boor – A. Pinkus, Proof of the conjectures of Bernstein and Erdős concerning the optimal nodes for polynomial interpolation, *J. Approximation Theory*, 24 (1978), 289-303.

P. Erdős – J. Szabados, On the integral of the Lebesgue function of interpolation, *Acta Math. Acad. Sci. Hungar.*, 32 (1978), 191-195.

P. Erdős – P. Vértesi, On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes, *Acta Math. Acad. Sci. Hungar.*, 36 (1980), 71-89; 38 (1981), 263.

P. Erdős, Problems and results on the theory of interpolation, I and II, *Acta Math. Acad. Sci. Hungar.*, 9 (1958), 381-388 and *ibid*, 12 (1961), 235-244.

P. Erdős, Some remarks on interpolation, *Acta Sci. Math. Szeged*, 12 (1950), 11-17.

P. Erdős – G. Grünwald, Über die arithmetische Mittelwerte der Lagrangeschen Interpolationspolynome, *Studia Math.*, 7 (1937), 82-95.

For a fruitful source of problems on interpolation and approximation see

P. Turán, Some open problems of approximation theory, *Mat. Lapok*, 25 (1-2), 21-75.

An English version of this paper will soon appear in the *Journal of Approximation Theory*.

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