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ON THE FAVOURITE POINTS OF A RANDOM WALK

P. Erdös, P. Révész

Dedicated to Prof. L. Iliev on the occasion of his 70th birthday

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of i. i. d. r. v.'s with $\mathsf{P}(X_1=+1)=\mathsf{P}(X_1=-1)=\frac{1}{2}$ and define the random walk $\{S(n)\}_{n=0}^{\infty}$ by $S(0)=0$, $S(n)=X_1+X_2+\dots+X_n$ $(n=1, 2, \dots)$. Consider also the r. v.'s $\xi(x, n)=\text{No.}\{k:\ k\leq n,\ S(k)=x\}$ $(x=0, \pm 1, \dots,\ n=1, 2, \dots)$ (where $\text{No}\{\dots\}$ is the cardinality of the indicated set) and $\xi(n)=\sup_x \xi(x, n)$.

The random set $\mathscr{F}_n=\{x:\ \xi(x, n)=\xi(n)\}$ will be called the set of favourite points of the random walk $\{S(n)\}$ at time n. The largest favourite points will be denoted by $f_n=\max\{x:\ x\in\mathscr{F}_n\}$.

In this paper we intend to study the properties of the random sequence $\{f_n\}$ and to formulate some unsolved problems on $\{\mathscr{F}_n\}$.

In order to formulate our results we repeat the definitions of the upper-lower classes by Lévy and remind the reader of the Erdös (1942) — Feller (1943-46), (1933-34) test [3, 4, 5].

Definition 1. The sequence $\{u(k)\}_{k=1}^{\infty}$ belongs to the upper class of $\{S(n)\}$ if

$$S(n)\leq u(n)\, n^{1/2},$$

except for finitely many n with probability 1.

Definition 2. The sequence $\{l(k)\}_{k=1}^{\infty}$ belongs to the lower class of $\{S(n)\}$ if

$$S(n)\geq l(n)\, n^{1/2}$$

infinitely often with probability 1.

Theorem A. *The increasing sequence $\{u(n)\}$ belongs to the upper class of $\{S(n)\}$ if and only if*

$$\Sigma\, n^{-1}u(n)e^{-u^2(n)/2}<\infty. \tag{1}$$

We remark that if $u_p(n)=(2L_2(n)+3L_3(n)+2L_4(n)+\dots+(2+\varepsilon)L_p(n))^{1/2}$ $(\varepsilon>0)$, then (1) holds true, but for the sequence $l_p(n)=(2L_2(n)+3L_3(n)+2L_4(n)+\dots+2L_p(n))^{1/2}$ we have

$$\Sigma\, n^{-1}\, l_p(n)e^{-l_p^2(n)/2}=\infty \quad (p=2, 3, \dots),$$

i. e. $\{u_p(n)\}$ belongs to the upper class and $\{l_p(n)\}$ belongs to the lower class for any $p=2, 3, \dots$ Here and in what follows

$$L_1(x)=\begin{cases} \log x & \text{if} \quad x\geq e \\ 1 & \text{if} \quad 0<x\leq e \end{cases}$$

and

$$L_p(x)=L_1(L_{p-1}(x)).$$

On the properties of $\{f_n\}$ as a trivial consequence of Theorem A one can see that $f_n\leq u(n)\,.\,n^{1/2}$ *with probability* 1 *except for finitely many* n *if* $\{u(n)\}$ *belongs to the upper class of* $\{S(n)\}$, *i. e. if* $\{u(n)\}$ *is increasing and* (1) *holds.*

Hence we have a trivial result saying that f_n cannot be very large. In our first theorem we prove that f_n occasionally will be large enough indeed. In fact we have:

Theorem 1. *For any* $\varepsilon>0$

$$f_n\geq((2-\varepsilon)nL_2(n))^{1/2}$$

with probability 1 *infinitely often.*

Having this result, one can conjecture that f_n will be larger than any function $l(n)\sqrt{n}$ i. o. with probability 1 if $l(n)$ belongs to the lower class of $\{S(n)\}$. However it is not the case. Conversely, we have

Theorem 2. $f_n\leq(n(2L_2(n)+3L_3(n)+2L_4(n)+2L_5(n)+2L_6(n)))^{1/2}$ *with probability* 1 *except for finitely many* n.

Theorem 1, resp. 2, will be proved in Section 2, resp. 3. In Section 4 we present a few unsolved problems on $\mathscr{F}_n$.

2. Proof of Theorem 1. Let $a_k=\exp(k^{1+\theta})$ $(\theta>0, k=1, 2, \dots)$ and introduce the notations:

$$\mathscr{A}(k)=\mathscr{A}(k, \theta, \varepsilon, \delta)=\{S([1-\varepsilon)a_{k+1}])-S([a_k])\geq(2(1-\delta)a_{k+1}L_2(a_{k+1}))^{1/2}\}$$

$$(0<\varepsilon<\delta<1, \quad \theta>0, \quad k=1, 2, \dots),$$

$$\mathscr{B}(k)=\mathscr{B}(k, \theta, \varepsilon, C)=\{\max_x(\xi(x, [(1-\varepsilon)a_{k+1}])-\xi(x, [a_k]))<Ca_{k+1}^{1/2}\}$$

$$(\theta>0, \quad C>0, \quad 0<\varepsilon<1, \quad k=1, 2, \dots),$$

$$\mathscr{C}(k)=\mathscr{C}(k, \theta, \varepsilon, D)=\{\xi(S([(1-\varepsilon)a_{k+1}]), [a_{k+1}])-\xi(S([(1-\varepsilon)a_{k+1}]),$$

$$[(1-\varepsilon)a_{k+1}])\geq(2Da_{k+1}L_2(a_{k+1}))^{1/2}\}$$

$$(\theta>0, \quad D>0, \quad 0<\varepsilon<1, \quad k=1, 2, \dots)$$

$$\mathscr{D}(k)=\mathscr{D}(k, \theta, \varepsilon, E)=\{S([1-\varepsilon)a_{k+1}])-\inf_{(1-\varepsilon)a_{k+1}\leq l\leq a_{k+1}} S(l)\geq E(\varepsilon a_{k+1})^{1/2}\}$$

$$(\theta>0, \quad E>0, \quad 0<\varepsilon<1, \quad k=1, 2, \dots).$$

Now we formulate a few lemmas.

Lemma 1. *There exists a positive constant* $\mathscr{K}$ *such that*

$$\mathsf{P}(\mathscr{A}(k))\geq Kk^{-\frac{1-\delta}{1-2\varepsilon}(1+\theta)}.$$

Proof is trivial.

Lemma 2. (cf. Kesten (1965)[2]). *For any* $\varepsilon>0$ *and* $\theta>0$ *there exists a constant* $C=C(\varepsilon, \theta)>0$ *such that* $\mathsf{P}(\mathscr{B}(k, \theta, \varepsilon, C))\geq 1/2$ $(k=1, 2, \dots)$.

The following lemma can be proved easily by the reader.

Lemma 3. *The conditional probability* $\mathsf{P}(\mathscr{B}(k)\,|\,S([(1-\varepsilon)a_{k+1}])-S([a_k])=y)$ *is an increasing function of* y $(y>0)$.

Lemma 4. (cf. Kesten (1965)[2]). *For any* $\theta>0, D>0, 0<\varepsilon<1$ *there exists a positive constant* $K=K(\theta,\ D,\ \varepsilon)$ *such that*

$$\mathsf{P}(\mathscr{C}(k))\geqq Kk^{-D/\varepsilon}\qquad (k=1,\ 2,\ldots).$$

A simple consequence of Lemma 4 is

Lemma 5. *For any* $\theta>0,\ D>0,\ E>0,\ 0<\varepsilon<1$ *there exists a positive constant* $K=K(\theta,\ D,\ E,\ \varepsilon)$ *such that*

$$\mathsf{P}(\mathscr{C}(k)\mathscr{D}(k))\geqq Kk^{-D/\varepsilon}\qquad (k=1,\ 2,\ldots).$$

Lemma 6. *For any positive* $\varepsilon,\ \theta,\ \delta$ *there exists a positive constant* $C=C(\varepsilon,\ \theta,\ \delta)$ *such that*

$$\mathsf{P}(\mathscr{B}(k,\ \theta,\ \varepsilon,\ C)\,|\,\mathscr{A}(k,\ \theta,\ \varepsilon,\ \delta))\geqq 1/2\quad (k=1,\ 2,\ldots).$$

Proof follows immediately from Lemmas 2 and 3.

Lemma 7. *For any* $\delta>0$ *one can find positive constants* $\varepsilon,\ \theta,\ C,\ D$ *and* K *such that*

$$\mathsf{P}(\mathscr{A}(k)\mathscr{B}(k)\mathscr{C}(k)\mathscr{D}(k))\geqq Kk^{-1}\quad (k=1,\ 2,\ldots).$$

Proof. We have

$$\mathsf{P}(\mathscr{A}(k)\mathscr{B}(k)\mathscr{C}(k)\mathscr{D}(k))=\mathsf{P}(\mathscr{A}(k)\mathscr{B}(k))\mathsf{P}(\mathscr{C}(k)\mathscr{D}(k))$$

$$=\mathsf{P}(\mathscr{B}(k)\mathscr{A}(k))\mathsf{P}(\mathscr{A}(k))\mathsf{P}(\mathscr{C}(k)\mathscr{D}(k))\geqq\frac{1}{2}\,Kk^{-\frac{1-\delta}{1-2\varepsilon}(1+\theta)}\,Kk^{-\frac{D}{\varepsilon}},$$

which proves the lemma.

Since the events $\mathscr{A}(k)\mathscr{B}(k)\mathscr{C}(k)\mathscr{D}(k)$ $(k=1,\ 2,\ldots)$ are mutually independent, Lemma 7 implies that with probability one infinitely many among them will occur. The event $\mathscr{A}(k)\mathscr{B}(k)\mathscr{C}(k)\mathscr{D}(k)$ implies that

$$\sup_x(\xi(x,\ a_{k+1})-\xi(x,\ a_k))\geqq(2Da_{k+1}L_2(a_{k+1}))^{1/2}$$

and if

$$\xi(x,\ a_{k+1})-\xi(x,\ a_k)=\sup_x(\xi(x,\ a_{k+1})-\xi(x,\ a_k)),$$

then $x\geqq(2(1-2\delta)a_{k+1}L_2(a_{k+1}))^{1/2}$.

Since $(2a_kL_2(a_k))^{1/2}=o((2a_{k+1}L_2(a_{k+1}))^{1/2}$, our Theorem 1 follows from the following

Lemma 8. (cf. Kesten (1965)[2]). *With probability one we have*

$$\limsup_{k\to\infty}\frac{\xi(a_k)}{(2a_kL_2(a_k))^{1/2}}\leqq 1.$$

3. Proof of Theorem 2. The following lemma can be obtained by a simple calculation.

Lemma 9. *For any* $i=0,\ 1,\ldots,[L_2(n)]-1$ *and* $p=2,\ 3,\ldots$ *we have*

$$\lim_{n\to\infty}P\{S([(i+1)\frac{n}{L_2(n)}])-S([i\frac{n}{L_2(n)}])\leqq y\,(\frac{n}{L_2(n)})^{1/2}\,|\,S_n\geqq(n(2L_2(n)+3L_3(n)$$

$$+2L_4(n)+\cdots+2L_p(n)))^{1/2}\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{(x-\sqrt{2})^2}{2}}\,dx \quad (-\infty<y<+\infty).$$

By a combinatorial argument one can see

Lemma 10. (Csáki-Földes (1983) [1]).

$$\mathsf{P}\{\xi(0,2k)=l\mid S(2k)=2m)=2^l\,\frac{(2m+l)\binom{2k-l}{k+m}}{(2k-l)\binom{2k}{k+m}}\quad (l=0,1,\ldots,k-m)$$

and

$$\lim_{n+\infty}\mathsf{P}\{\xi(0,n)<xn^{1/2}\mid S(n)=[yn^{1/2}]\}=1-\exp\left(-\frac{1}{2}((x+y)^2-y^2)\right)\quad (0\le x<\infty).$$

Given the sequence $S([i\frac{n}{L_2(n)}])$ $(i=0,1,2,\ldots,[L_2(n)]-1)$ the r. v.'s

$$\eta(i)=\xi(S([i\frac{n}{L_2(n)}]),\,[(i+1)\frac{n}{L_2(n)}])-\xi(S([i\frac{n}{L_2(n)}]),\,[i\frac{n}{L_2(n)}])$$

are clearly independent with distribution

$$\mathsf{P}\{\eta(i)<y\left(\frac{n}{L_2(n)}\right)^{1/2}\}=1-\exp\left(-\frac{1}{2}((y+\Delta_i)^2-\Delta_i^2)\right),$$

where

$$\Delta_i=(S([(i+1)\frac{n}{L_2(n)}])-S([i\frac{n}{L_2(n)}]))\left(\frac{L_2(n)}{n}\right)^{1/2}.$$

By a simple calculation again one gets

Lemma 11.

$$\mathsf{P}\{\max_{0\le i\le[L_2(n)]-1}\eta(i)\le K\left(\frac{n}{L_2(n)}L_4(n)\right)^{1/2}\mid \Delta_i=y_i,\quad i=0,2,\ldots,[L_2(n)]-1\}$$
$$\le C(L(n))^{-1},$$

where $K=K(y,y_2,\ldots,y_{[L_2(n)]-1})$ *is a big enough positive constant,* $C>0$ *is also big enough.*

Define the r. v. ν_n by $\nu_n=\inf\{k:\ S(k)\ge(n(2L_2(n)+3L_3(n)+2L_4(n)+2L_5(n))\}$ Theorem A implies

Lemma 12.

$$\nu_n\ge n\left(1-\frac{L_6(n)}{L_2(n)}\right)$$

with probability one except for finitely many n and

$$\mathsf{P}(\nu_n\le n)\le\frac{C}{L_1(n)L_2(n))^{3/2}L_3(n)L_4(n)L_5(n)}.$$

Applying again Kesten's result and Lemma 12, one gets

Lemma 13. *There exists a* $C>0$ *such that*

$$\mathsf{P}\{\sup_x(\xi(x,n)-\xi(x,\nu_n))\ge C\left(n\frac{L_5(n)}{L_2(n)}\right)^{1/2}\}\le\frac{C}{L_4(n)}.$$

ntroduce the following notations:

$$\mathscr{A}(n)=\{f_n\geq(n(2L_2(n)+3L_3(n)+2L_4(n)+L_5(n)))^{1/2}\},$$
$$\mathscr{B}(n)=\{\nu_n\leq n\},$$
$$\mathscr{C}(n)=\mathscr{C}(n,K)=\{\max_{0\leq i\leq[L_2(n)-L_6(n)]}\eta(i)\leq K(\frac{n}{L_2(n)}L_4(n))^{1/2}\},$$
$$\mathscr{D}(n)=\mathscr{D}(n,\ C)=\{\sup_x(\xi(x,\ n)-\xi(x,\ \nu_n))\geq C(n\frac{L_3(n)}{L_2(n)})^{1/2}\}.$$

Then we have

$$\mathscr{A}\subset[\mathscr{B}\cap(\mathscr{C}\cup\{\max_{0\leq i\leq[L_2(n)-L_6(n)]}\eta_i\leq\sup_x(\xi(x,\ n)-\xi(x,\ \nu_n))\})$$
$$\cup\ [\mathscr{B}\cap\overline{(\mathscr{C}}\cup\{\max_{0\leq i\leq[L_2(n)]-[L_6(n)]}\eta_i\leq\sup_x(\xi(x,n)-\xi(x,\nu_n))\})]\subset(\mathscr{B}\cap\mathscr{C})\cup(\mathscr{B}\cap\mathscr{D}).$$

Hence

$$\mathsf{P}(A)\leq\frac{C}{L_1(n)(L_2(n))^{3/2}L_3(n)(L_4(n))^2L_5(n)},$$

which implies that among the events $\mathscr{A}([a_k])$ (where $a_k=\exp(\frac{k}{(\log k)^{1/2}})$) only finitely many will occur with probability 1.

In fact the above proof gives a bit more:

Lemma 14. *Among the events*

$$\mathscr{A}^*(k)=\{\sup_{n\leq a_k}f_n\geq(a_k(2L_2(a_k)+3L_3(a_k)+2L_4(a_k)+2L_5(a_k)))^{1/2}\}$$

only finitely many can occur with probability 1.

This lemma and the trivial inequality

$$(a_k(2L_2(a_k)+3L_3(a_k)+2L_4(a_k)+2L_5(a_k)+2L_6(a_k)))^{1/2}$$
$$\geq(a_{k+1}(2L_2(a_{k+1})+3L_3(a_{k+1})+2L_4(a_{k+1})+2L_5(a_{k+1})))^{1/2}$$

implies our Theorem 2.

4. A Few Unsolved Problems. 1. The upper and lower estimates of f_n are far away from each other. It is not very hard to find somewhat better estimates, however a precise description of the upper and lower classes of f_n seems to be hard.

2. Our Theorem 1 stated that $f_n\geq((2-\varepsilon)nL_2(n))^{1/2}$ infinitely often with probability 1. Its proof shows that when $f_n\geq((2-\varepsilon)nL_2(n))^{1/2}$, then $\xi(f_n, n)=\xi(n)$ will be larger than $(2DnL_2(n))^{1/2}$ (where D is a small enough positive constant) infinitely often with probability one. As it is well known for any fixed x

$$\limsup_{n\to\infty}\frac{\xi(x,\ n)}{(2nL_2(n))^{1/2}}=1 \text{ with probability one.}$$

Suppose that for a random sequence $\{x_n\}$ we have

$$\limsup_{n\to\infty}\frac{\xi(x_n,\ n)}{(2nL_2(n))^{1/2}}=1.$$

Our question is: how big can x_n be?

3. Everyone can see immediately that No $\{\mathcal{F}_n\} \geqq 2$ infinitely often with probability one. Can we say that No $\{\mathcal{F}_n\} \geqq 3$ infinitely often with probability one?

4. Consider the random sequence ν_n for which No $\{\mathcal{F}_{\nu_n}\} \geqq 2$. What can we say about the sequence $\{\nu_n\}$? Can we say, for example, that $\lim_{n\to\infty} \nu_n/n = \infty$ with probability 1?

5. What are the properties of the sequence $|f_{n+1} - f_n|$? Is it true that $\limsup_{n\to\infty} |f_{n+1} - f_n| = \infty$? If yes, what is the rate of convergence?

6. Does the sequence $f_n/\sqrt{n}$ have a limit distribution? If yes, what is it?

7. Is it true that $0 \in \mathcal{F}_n$ infinitely often with probability one?

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Mathematical Institute of the Hungarian Academy of Sciences,
Reáltanoda u. 13-15 Budapest V

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