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SIZE RAMSEY NUMBERS INVOLVING MATCHINGS

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ABSTRACT

Let F, G and H be finite, simple and undirected graphs. The edges and number of edges of a graph F will be denoted by $E(F)$ and $|E(F)|$ respectively. A graph $F \to (G, H)$ if every 2-coloring (say red and blue) of $E(F)$ produces either a "red" G or a "blue" H. The size Ramsey number $\hat{r}(G, H) = \min\{|E(F)|: F \to (G, H)\}$. For $t \geq 1$, the graph consisting of t independent edges will be denoted by tK_2. In this paper, bounds and in some cases exact values will be calculated for $\hat{r}(tK_2, G)$ for various classical graphs G, for example, when G is either a small order graph, a path, a cycle, a complete graph or a complete bipartite graph. Asymptotic results are obtained for some graphs in which exact values could not be calculated.

1. INTRODUCTION

Let F, G and H be finite, simple and undirected graphs. The vertices and edges of a graph F will be denoted by $V(F)$ and $E(F)$ respectively. A graph $F \to (G, H)$ if every two coloring (say red and blue) of $E(F)$

produces either a red G or a blue H. The Ramsey minimal graphs for the pair (G, H) will be denoted by $R(G, H) = \{F\colon F \to (G, H)$ but $F' \nrightarrow (G, H)$ if $F' \subsetneqq F\}$. The generalized Ramsey number $r(G, H) = \min\{|V(F)|\colon F \in R(G, H)\}$ and the size Ramsey number $\hat{r}(G, H) = \min\{|E(F)|\colon F \in R(G, H)\}$.

The Ramsey number $r(G, H)$ has been calculated for many pairs of graphs (see [3] for a survey), but the size Ramsey number $\hat{r}(G, H)$ seems to be much more difficult to determine. Since its introduction in [6], there have been few results. The two numbers are of course related. For example it is clear that $\hat{r}(G, H) \leqslant \binom{r(G, H)}{2}$. In fact for complete graphs there is equality.

Theorem A (Chvatal [6]). *For* $n, m \geqslant 2$

$$r(K_n, K_m) = \binom{r(K_n, K_m)}{2}.$$

On the other hand for paths and cycles just the opposite occurs. Trivially $r(C_n, C_n) \geqslant r(P_n, P_n) > n$, *so* $\binom{r(P_n, P_n)}{2} > \frac{n^2}{2}$. *However there is the following result which answer a question of* Erdős (see [5] and [6]).

Theorem B (Beck [1]). *There is a constant* c *such that*

$$\hat{r}(P_n, P_n) \leqslant \hat{r}(C_n, C_n) \leqslant cn.$$

It is reasonable to start an investigation of size Ramsey numbers with pairs of graphs for which the generalized Ramsey number is known. The Ramsey numbers are known for most classical graphs when paired with a matching (see [7]). But even in this case there are many difficulties in calculating the size Ramsey number. In the second section of this paper exact values of $\hat{r}(tK_2, G)$ will be calculated when G is a graph of small order, a star, a large complete graph, or $t = 2$ and G is a complete bipartite graph. In the third section we will determine upper and lower bounds for $\hat{r}(tK_2, G_n)$ when n is large and G_n is a special graph on n vertices. In particular G_n a complete bipartite graph, path, cycle or a generalized wheel will be considered. In the last section results on

$\lim\limits_{t \to \infty} \dfrac{\hat{r}(tH, G)}{t\hat{r}(H, G)}$ will be given, with the emphasis on the case when $H = K_2$ and G is a classical graph. Notation not specifically mentioned will follow [8].

2. EXACT RESULTS

We start with some trivial but useful observations. If $F \to (G, H)$, then $tF \to (tG, H)$. Hence

$$(1) \qquad \hat{r}(tG, H) \leqslant t\hat{r}(G, H).$$

In particular, if $G = K_2$, then

$$(2) \qquad \hat{r}(tK_2, H) \leqslant t\,|E(H)|.$$

Also, since $r(tK_2, K_n) = n + 2t - 2$ [7]

$$(3) \qquad \hat{r}(tK_2, H) \leqslant \binom{|V(H)| + 2t - 2}{2}.$$

We shall see in this section that both the bound in (2) and in (3) can be exact for appropriate H. In fact each can be exact for H an appropriate complete graph K_n, but (3) can only be exact when $H = K_n$. The later is true since $K_{n+2t-3} \to (tK_2, K_n - e)$.

The first statament of the following theorem is a special case of a result in [4]. In the second statement, $K_{1,n} + e$ is a graph with a star with $n - 2$ edges attached to a triangle.

Theorem 1. *For* t, $n \geqslant 1$, $\hat{r}(tK_2, K_{1,n}) = tn$. *For* $t \geqslant 1$, $n \geqslant 2$, $\hat{r}(tK_2, K_{1,n} + e) = t(n + 1)$.

Proof. We prove only the second equality. It is sufficient to show that if $F \to (tK_2, K_{1,n} + e)$, then $|E(F)| \geqslant t(n + 1)$. For $t = 1$ this result is trivial, so assume $t \geqslant 2$ and proceed by induction on t. For $v \in V(F)$, $(F - v) \to ((t - 1)K_2, K_{1,n} + e)$ and so $|E(F - v)| \geqslant (t - 1)(n - 1)$. Hence if $d(v) \geqslant n + 1$, then $|E(F)| \geqslant n + 1 + |E(F - v)| \geqslant t(n + 1)$. Therefore we assume $\Delta(F) = n$.

Select vertices $v_1, v_2, \ldots, v_t$ such that v_i has maximal degree d_i in $G - \{v_1, v_2, \ldots, v_{i-1}\}$. Since $F - \{v_1, \ldots, v_{t-1}\} \geqslant K_{1,n} + e$, $d_i = n$

for all i, and $\{v_1, \ldots, v_t\}$ is an independent set of vertices. If $|E(F)| < t(n+1)$, then $H = f - \{v_1, v_2, \ldots, v_t\}$ has at most $t-1$ edges. Thus $H \not\to tK_2$ and $F - E(H)$ is a bipartite graph. This contradicts the fact that $F \to (tK_2, K_{1,n} + e)$, and so $|E(F)| \geqslant t(n+1)$.∎

The size Ramsey number $\hat{r}(tK_2, T_n)$ is not the same for all trees T_n on n vertices. Theorem 1 and the following theorem verify that. Here $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Theorem 2. *For* $t \geqslant 1$, $\hat{r}(tK_2, P_4) = \lceil \frac{5t}{2} \rceil$ *and*

$$\hat{r}(tK_2, P_5) = \begin{cases} 3t & t \text{ even} \\ 3t+1 & t \text{ odd.} \end{cases}$$

Proof. For the first equality note that $C_5 \to (2K_2, P_4)$. Thus

$$\frac{t}{2} C_5 \to (tK_2, P_4) \qquad t \text{ even}$$

$$\frac{t-1}{2} C_5 \cup P_4 \to (tK_2, P_4) \qquad t \text{ odd.}$$

It remains to show that if $F \in R(tK_2, P_4)$, then $|E(F)| \geqslant \lceil \frac{5t}{2} \rceil$. For $t = 1, 2$ this is trivial. Therefore we assume $t \geqslant 3$, and proceed by indunction on t. For $v \in V(G)$, $(F - v) \to ((t-1)K_2, P_4)$. Hence if $d(v) \geqslant 3$, $|E(F)| \geqslant 3 + |E(G-v)| \geqslant 3 + \lceil \frac{5(t-1)}{2} \rceil \geqslant \lceil \frac{5t}{2} \rceil$. Thus we assume $\Delta(F) = 2$, and F is the union of paths and cycles each of which must contain a P_4. In this case one can delete an appropriate two vertices from F and then the edges not on a P_4 to obtain a graph F' such that $F' \to ((t-2)K_2, P_4)$ and $|E(F')| \leqslant |E(F)| - 5$. Hence using the induction assumption, $|E(F)| \geqslant 5 + |E(F')| \geqslant \lceil \frac{5t}{2} \rceil$.

For the second equality, note that $C_6 \to (2K_2, P_5)$. A repeat of the argument for P_4 gives the second equality.∎

It is not true that $\hat{r}(tK_2, P_n)$ is approximately $\frac{t(n+1)}{2}$ for arbitrary n. Theorem B implies that there is a constant c such that for t sufficiently large, $\hat{r}(tK_2, P_n) \leqslant ct$. It would be nice to know how long this behavior does last.

Theorem 3. *For* $t \leqslant 1$ *and for all connected graphs* G *with* $|V(G)| \leqslant 4$ *except for* P_4,

$$\hat{r}(tK_2, G) = t|E(G)|.$$

Proof. The graphs we are concerned with are K_2, $K_{1,2}$, K_3, $K_{1,3}$, C_4, $K_3 + e$, $K_4 - e$ and K_4. The result is trivial for K_2 and it follows from Theorem 1 for K_3, $K_{1,3} + e$ and the stars. Because of (2), it is sufficient to show that if $F \in R(tK_2, G)$, then $|E(F)| \geqslant t|E(G)|$.

Each graph requires a separate argument, but there are some general observations that can be made. The result is trivial for $t = 1$, so we can assume $t \geqslant 2$, and proceed by induction on t. Each edge of F must be in a copy of G. Thus $\delta(F) \geqslant \delta(G)$. This also implies that if $F' \to (sK_2, G)$, then $(F' - e) \to (sK_2, G)$ if $e \in E(F')$ is not in a copy of G. For $v \in V(F)$, $(F - v) \to ((t-1)K_2, G)$. Thus, if $d(v) \geqslant |E(G)|$, the induction assumption implies that $|E(F)| \geqslant |E(G)| + |E(F - v)| \geqslant$ $\geqslant |E(G)| + (t-1)|E(G)|$. Therefore we can assume $\Delta(F) < |E(G)|$. There is no loss of generality in assuming F is connected. To complete the induction it is sufficient to find an $F' \subset F$ such that $F' \to ((t-1)K_2, G)$ and

$$(4) \qquad |E(F')| \leqslant |E(F)| - |E(G)|.$$

We will only exhibit one representative case $(G = C_4)$, and leave the remaining ones to the reader. If $G = C_4$, then $\delta(F) \geqslant 2$, $\Delta(F) \leqslant 3$ and we can assume F is connected. If $\Delta(F) = 2$, then the connectivity of F implies that $F = C_4$, so $\Delta(F) = 3$. If $\delta(F) = 2$, then there is an edge $uv \in E(F)$ with $d(u) = 2$ and $d(v) = 3$. Hence $(F - u - v) \to$ $\to ((t-1)K_2, C_4)$, since u had degree 1 in $F - v$. By (4) the proof is complete in this case. We can thus assume F is 3-regular. Let u_1, u_2, u_3, u_4 be consecutive vertices on a 4-cycle $C \leqslant F$. If C contains two chords, then $F = K_4$, which clearly gives a contradiction. If C contains no chords, then $F' = (F - u_1 - u_3 - u_2 - u_4) \to ((t-2)K_2, G)$ and $|E(F)| \geqslant 8 + |E(F')|$. An induction argument will complete the proof in this case. We can thus assume every 4-cycle in F contains precisely one chord. Since every edge of F is on a 4-cycle, it is straightforward to check that this is impossible, which completes the proof. ∎

The following theorem shows that the nature of $\hat{r}(tK_2, K_n)$ is different when n is large relative to t.

Theorem 4. *If* $t \geqslant 1$ *and* $n \geqslant 4t-1$, *then*

$$\hat{r}(tK_2, K_n) = \binom{n+2t-2}{2}.$$

The following technical lemma will be needed in the proof of the theorem.

Lemma 5. *If for* $s \geqslant 0$, H *is a graph on* m *vertices with* $m \geqslant \frac{5s+3}{2}$ *and* $|E(H)| > (m-s)s + \binom{s}{2}$, *then* $H \geqslant (s+1)K_2$.

Proof. If $H \not\geqslant (s+1)K_2$, then a well known theorem of Tutte ([2]) implies that

$$|E(H)| \leqslant \max\left\{\binom{k}{2} + \sum_{i=1}^{r}\binom{2s_i+1}{2} + (m-k)k\colon\ 0 \leqslant k \leqslant s\right.$$

$$\left.\sum_{i=1}^{r} s_i = s-k\right\}.$$

This reduces to

$$|E(H)| \leqslant \max\left\{\binom{k}{2} + \binom{2(s-k)+1}{2} + (m-k)k\colon\ 0 \leqslant k \leqslant s\right\} \leqslant$$

$$\leqslant \max\left\{\binom{s}{2} + s(m-s), \binom{2s+1}{2}\right\}.$$

It is straightforward to check that $\binom{s}{2} + s(m-s) \geqslant \binom{2s+1}{2}$ if $m \geqslant \frac{5s+3}{2}$.

Proof of Theorem 4. Since $r(tK_2, K_n) = n+2t-2$, $\hat{r}(tK_2, K_n) \leqslant \binom{n+2t-2}{2}$. To prove the theorem it is sufficient to show that if $F \in R(tK_2, K_n)$, then $|E(F)| \geqslant \binom{n+2t-2}{2}$. Assume F has

$n+2t-2+x$ vertices $(x \geqslant 0)$. Every vertex of F must be in a K_n, thus $\delta(F) \geqslant n-1$. If $x \geqslant \frac{(n+2t-2)(2t-2)}{n-1}$, then $|E(F)| \geqslant \geqslant \frac{(n+2t-2+x)(n-1)}{2} \geqslant \binom{n+2t-2}{2}$. We thus assume

$$x \leqslant \frac{(n+2t-2)(2t-2)-1}{n-1}. \tag{5}$$

It is straightforward to verify that (5) implies that $n+2t-2+x \geqslant \geqslant \frac{5x+3}{2}$. If

$$|E(\bar{F})| \geqslant \binom{x}{2} + x(n+2t-2) + 1, \tag{6}$$

then Lemma 5 gives that $\bar{F}$ (complement of F) contains a $(x+1)K_2$. Select $t-1$ vertex disjoint triangles in $F \cup \bar{F}$ which are also vertex disjoint from $(x+1)K_2$. Inequality (5) and $n \geqslant 4t-1$ assure that this can be done. Color the edges of F in the triangles red and color the remaining edges of F blue. There is no red tK_2 and the largest blue complete graph has at most $(n+2t-2+x)-(x+1+2t-2)=n-1$ vertices. Thus inequality (6) is not satisfied and $|E(F)| \geqslant \binom{n+2t-2}{2}$. ∎

Although the condition $n \geqslant 4t-1$ is not sharp, it cannot be removed from the conditions of the theorem. For example the following is true for $t=2$.

Theorem 6.

$$\hat{r}(2K_2, K_n) = \begin{cases} \binom{n+2}{2} & n \geqslant 6 \\ 2\binom{n}{2} & 2 \leqslant n \leqslant 5. \end{cases}$$

The above theorem is an immediate consequence of Theorem 3 and Theorem 4 except for $n=5, 6$. These two cases can easily be handled with the methods used in Theorems 3 and 4.

Theorem 7. *For* $n \geqslant 2$, $\hat{r}(2K_2, K_{n,n}) = n^2 + 2n$.

Later in the paper, Theorem 12 gives bounds for $\hat{r}(tK_2, K_{n,n})$ for n large. A slightly more careful analysis when $t=2$ gives Theorem 7, so we omit the proof. Also in a similar fashion some additional case analysis will give the following:

Theorem 8. *For* $n \geqslant m \geqslant 2$, $\hat{r}(2K_2, K_{n,m}) = nm + n + m$.

3. BOUNDS

The difficulty in calculating $\hat{r}(tK_2, G)$ when G is just a P_n, C_n or a $K_{n,n}$ is surprising. In this section reasonable upper and lower bounds giving the magnitude of the size Ramsey number will be determined. First some lemmas will be proved.

Lemma 8. *For a fixed* $t \geqslant 2$, *there is a constant* c *(depending on* t*) such that*

$$\hat{r}(tK_2, C_n) < n + c\sqrt{n}.$$

Proof. For small values of n, appropriate choice of c will imply the result. We can thus assume that n is large. Let $l = [\sqrt{n}]$. Consider the graph C^t_{l+2t} (a cycle of length $l + 2t$ containing all chords of length $\leqslant t$). Replace each edge on this cycle with a path of length l, to obtain a graph H. Denote the vertices of this new cycle by $\{x_0, x_1, \dots, x_{l^2+2tl-1}\}$. Let $G(n, t)$ be the graph obtained from H by the addition of the edges $\{x_{il}, x_{il+j}\colon 0 \leqslant i < t,\ 2 \leqslant j < l\}$.

For any set $S \leqslant V(H) = V(G(n, t))$ with $|S| = t - 1$, it is easily seen that $H - S$ contains a cycle of length r for some r, $n \leqslant r < n + l$, and that $G(n, t)$ contains a cycle of length n. It follows immediately that $r(tK_2, C_n) \leqslant |E(G(n, t))|$. Since

$$|E(G(n, t))| < l^2 + 2lt + (l + 2t)t + t(l - 2) \leqslant n + (4t + 3)\sqrt{n},$$

the proof is complete.∎

Lemma 9. *If* F *is a graph such that* $F - v_1 - v_2 \geqslant P_n$ *for any pair* $v_1, v_2 \in V(F)$, *then there is a positive constant* c *such that* $|E(F)| \geqslant$ $\geqslant n + c\sqrt{n}$.

Proof. For small values of n an appropriate choice of c implies the result, so we assume n is large. Assume F has m vertices and $m + r$ edges. Let H be the graph obtained from F by contracting each suspended path (path, whose interior vertices have degree 2) to an edge. Thus H has no vertices of degree 2. Let l be the number of vertices of degree $\geqslant 3$ and s the number of vertices of degree 1. So H has the degree sequence $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_l > 1 \geqslant 1 \geqslant \ldots \geqslant 1$.

Any path contains at most 2 edges from any vertex and at most 2 vertices of degree 1. Thus at least $\dfrac{\sum_{i=1}^{l} (d_i - 2) + s - 2}{2}$ edges are not in any path. Since $F \geqslant P_n$, $l \leqslant \sum_{i=1}^{l} (d_i - 2) + s \leqslant 2(m + r - n + 2)$. No suspended path of F can contain $m + r - n$ edges, for otherwise the deletion of two appropriate vertices would leave two disjoint graphs with less than $n - 1$ edges. Therefore $|E(F)| \leqslant (m + r - n)|E(H)|$. This implies

$$\begin{aligned} m + r = |E(F)| &\leqslant (m + r - n)|E(H)| = \\ &= (m + r - n)\frac{\left(\sum_{i=1}^{l} d_i\right) + s}{2} \leqslant \\ &\leqslant (m + r - n)(m + r - n + 2 + l) < 3(m + r - n + 2)^2. \end{aligned}$$

Hence for $x = m + r - n + 2$, we have $x > \sqrt{\frac{n}{3}}$ and $m + r > n + \sqrt{\frac{n}{3}} - 2$, which completes the proof.∎

A very similar proof yields the following

Lemma 10. *If F is a graph such that $F - v \geqslant C_n$ for any $v \in V(F)$, then there is a constant c such that $|E(F)| \geqslant n + c\sqrt{n}$.*

The following is a direct consequence of Lemmas 8, 9 and 10.

Theorem 11. *For fixed t there are positive constants c_i $(1 \leqslant i \leqslant 4)$ such that for all $n \geqslant 3$,*

$$n + c_1 \sqrt{n} < \hat{r}(tK_2, C_n) < n + c_2 \sqrt{n} \qquad (t \geqslant 2)$$

and

$$n + c_3 \sqrt{n} < \hat{r}(tK_2, P_n) < n + c_4 \sqrt{n} \qquad (t \geqslant 3).$$

The restriction $t \geqslant 3$ is necessary for paths. Since $C_{n+1} \to (2K_2, P_n)$, $\hat{r}(2K_2, P_n) = n + 1$.

If G and H are graphs, then $G + H$ is the graph with disjoint copies of G and H and all edges between $V(G)$ and $V(H)$. For example $K_1 + C_n$ is generally called a wheel with n spokes.

Theorem 12. *For fixed* $s \geqslant 1$ *and* t, *there are positive constants* c_i $(1 \leqslant i \leqslant 4)$ *such that for all* n

$$(s + t)n + c_1 \sqrt{n} \leqslant \hat{r}(tK_2, \bar{K}_s + C_n) \leqslant (s + t)n + c_2 \sqrt{n} \quad (t \geqslant 2)$$

$$(s + t)n + c_3 \sqrt{n} \leqslant \hat{r}(tK_2, \bar{K}_s + P_n) \leqslant (s + t)n + c_4 \sqrt{n} \quad (t \geqslant 3).$$

Proof. Both upper bounds come from the fact that $F = \bar{K}_{s+t-1} + G(n, 2t) \to (tK_2, \bar{K}_s + C_n)$, where $G(n, 2t)$ is the graph defined in Lemma 8.

We now verify the first lower bound. Assume $F \to (tK_2, \bar{K}_s + C_n)$. Select vertices $A = \{v_1, v_2, \dots, v_{t-1}\} \leqslant V(F)$ such that v_i has maximal degree in $F - \{v_1, \dots, v_{i-1}\}$. Since $F - A \geqslant \bar{K}_s + C_n$, $d(v_i) \geqslant n$ $(1 \leqslant i \leqslant t - 1)$, so F has at least $s + t - 1$ vertices of degree n and a disjoint C_n. If n is small, an appropriate choice of the constant implies the result, so we can assume n is large. Also, if F has $s + t$ vertices of degree $\geqslant n$, then clearly the result follows.

Thus we assume that F has precisely $s + t - 1$ vertices of degree $\geqslant n$, which we denote by B. Let $H = F - B$. To complete the proof, it is sufficient to show that $|E(H)| \geqslant n + c\sqrt{n}$ for some constant $c > 0$. Select two disjoint sets B_1 and B_2 of B containing $t - 2$ and 2 vertices respectively, and let C be a single vertex of H. Color red all edges of F incident to a vertex of B_1 as well as any edge with both end vertices in $B_2 \cup C$. Color the remaining edges blue. There is no red tK_2, so there must be a blue $\bar{K}_s + C_n$. Hence H must contain a blue C_n

avoiding the vertex in C. But Lemma 10 implies that $|E(H)| \geqslant n + c\sqrt{n}$.

The exact same proof is valid for the second lower bound, except that B_1, B_2 and C have $t-3$, 3 and 2 vertices respectively in this argument. ∎

We end this section with a result on the complete bipartite graph $K_{n,n}$ mentioned earlier.

Theorem 13. *For $t \geqslant 0$ and n sufficient large,*

$$n^2 + 2tn - \binom{2t}{2} \leqslant \hat{r}((t+1)K_2, K_{n,n}) \leqslant n^2 + 2tn + \binom{t}{2}.$$

Proof. For $t = 0$, the result is trivial and the bounds agree. Assume $t \geqslant 1$. We first verify the upper bound. Let H be the symmetric bipartite graph with $2t$ vertices such that each of the vertices in each part have distinct non-zero degrees. Thus H is unique and $|E(H)| = \binom{t+1}{2}$. Consider H as a subgraph of $K_{n+t,n+t}$ and let $F = K_{n+t,n+t} - E(H)$. It is straightforward to verify that the deletion of t vertices from F leaves a $K_{n,n}$. Therefore $F \to ((t+1)K_2, K_{n,n})$ and $|E(F)| \leqslant n^2 + 2tn + \binom{t}{2}$.

To verify the lower bound we will assume that

$$F \in R((t+1)K_2, K_{n,n}) \quad \text{and} \quad E(F) < n^2 + 2nt - \binom{2t}{2},$$

and we will show that this leads to a contradiction. Each edge of F must be in a $K_{n,n}$, so $\delta(F) \geqslant n$ and $F \geqslant K_{n,n}$. If $|V(F)| \geqslant 2n + 2t + 1$, then $|E(F)| \geqslant n^2 + (2t+1)n - \binom{2t+1}{2} \geqslant n^2 + 2tn - \binom{2t}{2}$ for large n. We thus have $|V(F)| \leqslant 2n + 2t$.

Since $r((t+1)K_2, K_{n,n}) = 2n + t$, $|V(F)| = 2n + t + s$ for $0 \leqslant s \leqslant t$. Let D denote the $t-s$ vertices of F of largest degree. The graph $F - D \geqslant K_{n,n}$. Let A and B be the parts of this bipartite graph and $C = V(F) - (A \cup B \cup D)$. Thus $|C| = 2s$. No vertex of $A \cup B \cup C$ has degree $\geqslant 2n - 5t$, for if so, then every vertex in D has degree $\geqslant 2n - 5t$. This would imply

$$|E(F)| \geqslant n^2 + 2sn + (t-s)(2n-5t) + n - 5t - \binom{t+s}{2} =$$

$$= n^2 + 2nt + (n - (t-s+1)5t) - \binom{t+s}{2},$$

which for n large gives a contradiction.

Let a be the number of edges between vertices of A. Likewise define b. Note that $n^2 + a + (t+s)n - \binom{t+s}{2} \leqslant n^2 + 2nt - \binom{2t}{2}$, so $a \leqslant (t-s)n$. We will use this to show that $a = 0$. Assume not, and let xy be such an edge. There is a $K_{n,n}$ with parts X and Y with $x \in X$ and $y \in Y$. Degree restrictions on x and y imply that $|X \cap A|, |Y \cap A| \leqslant n - 5t$. With no loss of generality we can assume $|X \cap A| \geqslant 3t$ and $|Y \cap A| \geqslant \frac{n-2t}{2}$. Therefore $a \geqslant \frac{3t(n-2t)}{2}$, which contradicts the previous restriction on a. Hence $a = b = 0$.

Each vertex $c \in C$ is incident to an edge which is in a $K_{n,n}$. Since all of the vertices of A (B) must be in the same part of a bipartite graph, the vertex c must have $|\Gamma(c) \cap A| \geqslant n - 2t$ or $|\Gamma(c) \cap B| \geqslant n - 2t$. Both inequalities cannot be satisfied. Thus C can be partitioned into $C_A \cup C_B$, where $C_M = \{c \in C\colon |\Gamma(c) \cap M| \geqslant n - 2t\}$. Let $A' = A \cup C_A$ and $B' = B \cup C_B$. The argument of the previous paragraph also implies that there are no edges between vertices of A' (or B').

Let $T \leqslant A'$ with $|T| = t$. Thus $F - T \geqslant K_{n,n}$. For this to occur, $|A'| - t + |D| \geqslant n$. Hence $|A'| \geqslant n + s$ and likewise $|B'| \geqslant n + s$. This implies $|A'| = |B'| = n + s$. Therefore $F - T \geqslant K_{n,n}$ requires that each vertex of D must be adjacent to at least n vertices in B (and also in A). Thus $d(v) \geqslant 2n$ for $v \in D$, and $|E(G)| \geqslant$ $\geqslant n^2 + 2sn + (t-s)2n - \binom{t+s}{2}$. This gives a contradiction, which completes the proof.∎

4. ASYMPTOTIC RESULTS

In this section we will consider for fixed nontrivial graphs H and G,

$$\lim_{t\to\infty} \frac{\hat{r}(tH, G)}{t\hat{r}(H, G)}.$$

Equation (1) implies that the above ratio is $\leqslant 1$. The following theorem is also a consequence of (1).

Theorem 14. $\lim_{t\to\infty} \frac{\hat{r}(tH, G)}{t\hat{r}(H, G)}$ *exists.*

Proof. Let $a_t = \frac{\hat{r}(tH, G)}{t\hat{r}(H, G)}$ and a be the greatest lower bound of $\{a_t\colon t \geqslant 1\}$. For any $\epsilon > 0$ there is an m such that $a \leqslant a_m < a + \frac{\epsilon}{2}$. Select l such that $\frac{a_m}{l} < \frac{\epsilon}{2}$ and let $N = lm$. If $n \geqslant N$ then $n = qm + r$ with $q \geqslant l$ and $0 \leqslant r < m$. Thus

$$\begin{aligned} a_n &= \frac{\hat{r}((qm + r)H, G)}{(qm + r)\hat{r}(H, G)} \leqslant \frac{\hat{r}((q + 1)mH, G)}{qm\hat{r}(H, G)} \leqslant \\ &\leqslant \frac{q + 1}{q}\, a_m \leqslant \frac{l + 1}{l}\, a_m < a + \epsilon. \blacksquare \end{aligned}$$

Question. *Which numbers in* $[0, 1]$ *can be limit points of a sequence* $\left(\frac{\hat{r}(tH, G)}{t\hat{r}(H, G)}\right)_{t=1}^{\infty}$?

The observation that $\hat{r}(tH, G) \geqslant t\,|E(H)|$ implies that

$$\frac{\hat{r}(tH, G)}{t\hat{r}(H, G)} \geqslant \frac{|E(H)|}{\hat{r}(H, G)} > 0. \tag{7}$$

Therefore 0 is not a limit point. It is possible for 1 to be a limit point, but of course

$$\lim_{t\to\infty} \frac{\hat{r}(tH, G)}{t\hat{r}(H, G)} = 1, \quad \text{iff} \quad \hat{r}(tH, G) = t\hat{r}(H, G) \quad \text{for all} \quad t.$$

We now consider the case when $H = K_2$ and denote the

$$\lim_{t\to\infty} \frac{r(tK_2, G)}{t\hat{r}(K_2, G)}$$

by $\hat{r}_\infty(G)$. Theorems 1 and 3 can be restated as follows:

Theorem 15. *If* G *is a* $K_{1,n}$, $K_{1,n}+e$, $n \geqslant 2$, *or a connected graph with* $|V(G)| \leqslant 4$ *which is not a* P_4, *then* $\hat{r}_\infty(G) = 1$.

Question. Is there an infinite family of graphs $\{G_n\colon n \geqslant 1\}$ with $|V(G_n)| = n$ and $|\Delta(G_n)| < n-1$ such that

$$\hat{r}_\infty(G) = 1?$$

On the other hand for paths, cycles, complete bipartite graphs and complete graphs, $\hat{r}_\infty$ assumes small values

Theorem 16. *For any* $n \geqslant 2$, *there is a positive constant* a *such that*

$$\frac{1}{n} \leqslant \hat{r}_\infty(P_n), \quad \hat{r}_\infty(C_n) \leqslant \frac{a}{n}$$

$$\frac{1}{n} \leqslant r_\infty(K_{n,n}) \leqslant \frac{2+\sqrt{2}}{n}, \quad \frac{2}{n} \leqslant r_\infty(K_n) \leqslant \frac{8}{n}.$$

Proof. The first lower bound follows directly from inequality (7). Note that Theorem B implies that $\hat{r}(nK_2, P_{2n}) \leqslant \hat{r}(nK_2, C_{2n}) \leqslant \leqslant \hat{r}(C_{2n}, C_{2n}) \leqslant 2cn$. The first upper bound follows immediately.

If $F \in R(tK_2, K_{n,n})$, then $\Delta(F) \geqslant n$ and thus $|E(F)| \geqslant tn$. Hence

$$\frac{\hat{r}(tK_2, K_{n,n})}{t\hat{r}(K_2, K_{n,n})} \geqslant \frac{1}{n}.$$

On the other hand the proof of the upper bound in Theorem 13 gives that $\hat{r}(tK_2, K_{n,n}) \leqslant n^2 + 2(t-1)n + \binom{t-1}{2}$. If we let $t = [\sqrt{2}\,n]$, then

$$\frac{\hat{r}(tK_2, K_{n,n})}{tn^2} \leqslant \frac{2+\sqrt{2}}{n}.$$

This verifies the second set of inequalities.

If $F \to (tK_2, K_n)$, then $\Delta(F) \geqslant n-1$. Thus $|E(F)| \geqslant t(n-1)$ and

$$\frac{\hat{r}(tK_2, K_n)}{t\hat{r}(K_2, K_n)} \geqslant \frac{t(n-1)}{t\binom{n}{2}} = \frac{2}{n}.$$

Also $K_{n+2t-2} \to (tK_2, K_n)$. If we let $t = \lceil \frac{n}{2} \rceil$, then

$$\hat{r}_\infty(K_n) \leqslant \frac{\binom{n + 2\lceil \frac{n}{2} \rceil - 2}{2}}{\lceil \frac{n}{2} \rceil \binom{n}{2}} \leqslant \frac{8}{n},$$

which completes the proof.∎

There are several interesting questions concerning $\hat{r}_\infty(K_n)$. Since $K_{n+2t-2} \to (tK_2, K_n)$,

$$\hat{r}_\infty(K_n) \leqslant \frac{\binom{n+2t-2}{2}}{t\binom{n}{2}}.$$

Let

$$M_n = \min \left\{ \frac{\binom{n+2t-2}{2}}{t\binom{n}{2}} : t \geqslant 1 \right\}.$$

Question. *Does* $\hat{r}_\infty(K_n) = M_n$?

The small values of M_n are $M_2 = M_3 = M_4 = M_5 = 1$ and $M_6 = \frac{14}{15}$. If the answer to the previous question is yes, then we should have $\hat{r}_\infty(K_2) = \hat{r}_\infty(K_3) = \hat{r}_\infty(K_4) = \hat{r}_\infty(K_5) = 1$ and $\hat{r}_\infty(K_6) = \frac{14}{15}$. The first three equalities are a consequence of Theorem 15.

Question. *Does* $\hat{r}_\infty(K_5) = 1$ *and* $\hat{r}_\infty(K_6) = \frac{14}{15}$?

The calculation of $\hat{r}_\infty(G)$ for some special graphs indicates that the values of $\hat{r}_\infty$ tend to be near 0 or 1. This is not in fact true, as the following theorem indicates.

Theorem 17. *The set* $\{\hat{r}_\infty(G)$: *G a connected graph*$\}$ *is dense in* $[0, 1]$.

Proof. Let $k \geqslant 2$, n and $l = l_1 \geqslant l_2 \geqslant \ldots \geqslant l_k \geqslant 0$ be integers such that $\sum_{i=1}^{k} l_i + 2k = n$. Then $H(n, k, l)$ will denote the graph containing a complete bipartite graph $K_{k,k}$ with disjoint stars of $l_1, l_2, \ldots, l_k$ edges respectively attached to the vertices in one part of the $K_{k,k}$. Thus $H(n, k, l)$ has n vertices and $k^2 + \sum_{i=1}^{k} l_i = n + k(k-2)$ edges.

Assume $F \to (tK_2, H(n, k, l))$. Select vertices $\{v_1, v_2, \ldots, v_{t-1}\}$ such that v_i has maximal degree d_i in $F - \{v_1, v_2, \ldots, v_{i-1}\}$. Since $F - \{v_1, \ldots, v_{t-1}\} \geqslant H(n, k, l)$, $d_i \geqslant l + k$ and $|E(F)| \geqslant$ $\geqslant (t-1)(l+k) + n + k(k-1)$.

Consider the graph G which contains a $K_{k+t-1,k+t-1}$ with $t-1$ disjoint stars of l edges and k disjoint stars of $l_1, l_2, \ldots, l_k$ edges respectively attached to the vertices in one part of the $K_{k+t-1,k+t-1}$. The graph G has

$$(k + t - 1)^2 + (t-1)l + \sum_{i=1}^{k} l_i =$$

$$= n + (t-1)l + (k + t - 1)^2 - 2k$$

edges. Also if $S \leqslant V(G)$ with $|S| = t - 1$, it is easy to see that $G - S \geqslant$ $\geqslant H(n, k, l)$. Therefore $G \to (tK_2, H(n, k, l))$ and $\hat{r}(tK_2, H(n, k, l)) \leqslant$ $\leqslant n + (t-1)l + (k + t - 1)^2 - 2k$.

For n and l large (relative to k and t),

$$\frac{\hat{r}(tK_2, H(n, k, l))}{t\,|E(H(n, k, l))|} \leqslant \frac{n + tl}{tn} \leqslant \frac{l}{n} + \frac{1}{t}$$

and

$$\frac{\hat{r}(tK_2, H(n, k, l))}{t\,|E(H(n, k, l))|} \geqslant \frac{n + (t-1)l}{(t+1)n} \geqslant \frac{l}{n} - \frac{1}{t+1}.$$

Thus $\frac{l}{n} - \frac{1}{t+1} \leqslant \hat{r}_\infty(H(n, k, l)) \leqslant \frac{l}{n} + \frac{1}{t}$. This completes the proof. ∎

The graphs appearing in Theorem 17 are sparse graphs. The following result shows that sparse graphs are the only possible graphs that could be used in the proof of the previous theorem.

Theorem 18. *Let* $\{G_n\colon n \geqslant 2\}$ *be a family of graphs with* $|V(G_n)| = n$. *If*

$$\lim_{n\to\infty} \frac{|E(G_n)|}{n} = \infty,$$

then

$$\lim_{n\to\infty} \hat{r}_\infty(G_n) = 0.$$

Proof. Since $r(nK_2, K_n) \leqslant 3n - 2$,

$$\lim_{n\to\infty} \hat{r}_\infty(G_n) \leqslant \lim_{n\to\infty} \frac{\hat{r}(nK_2, G_n)}{n|E(G_n)|} \leqslant \lim_{n\to\infty} \frac{\binom{3n-2}{2}}{n|E(G_n)|} \leqslant$$

$$\leqslant \frac{9}{2} \lim_{n\to 0} \frac{n}{|E(G_n)|} = 0.$$

A constant c such that $|E(G_n)| \leqslant cn$ does not imply that $\lim\limits_{n\to\infty} \hat{r}_\infty(G_n) > 0$. For example for paths and cycles the value of the limit is 0. It appears that G_n needs to be sparse and have large $\Delta(G_n)$ for the limit to be > 0.

Question. *If* $\{G_n\colon n > 1\}$ *is a family of graphs with* $|V(G_n)| = n$ *and* $\Delta(G_n) \leqslant c$, *is*

$$\lim_{n\to\infty} \hat{r}_\infty(G_n) = 0?$$

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