A Conjecture on Dominating Cycles

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ABSTRACT

A *dominating cycle* in a graph is a cycle in which every vertex of the graph is adjacent to at least one vertex on the cycle. We conjecture that for each c there is a constant k_c such that every c-connected graph with minimum degree $\delta \geq \frac{n}{c+1} + k_c$ has a dominating cycle. We show that this conjecture, if true, if best possible. We further prove the conjecture for graphs of connectivities 1 through 5.

1. Introduction

For notation, usually we follow Bondy and Murty [1]. The number of vertices, the connectivity and the minimum degree are denoted by n, c and δ, respectively. A *dominating cycle* is a cycle L in graph G for which every vertex of G is adjacent to at least one vertex of L. A more specific type of cycle is a *D-cycle* , which is a cycle L in graph G for which every edge of G is incident to at least one vertex of L.

Dominating cycles have been studied from an algorithmic viewpoint [3, 4 and 7] with applications in network design in mind. We are interested here instead in studying an extremal problem, namely the minimum degree which ensures that a c-connected graph contains a dominating cycle. Our primary motivation is **not** algorithmic, but rather to extend previous research on D-cycles and Hamilton cycles. A D-cycle can be considered as a generalization of a Hamilton cycle and a dominating cycle a generalization of a D-cycle. Therefore the smallest minimum degree that guarantees a dominating

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cycle should be smaller than that for a D-cycle, which in turn should be smaller than the sufficiency condition with respect to δ for Hamilton cycles.

Dirac's classical result gives the sufficiency condition with respect to δ for Hamiltonicity [5].

Theorem A1. Let G be a graph with $n \geq 3$ and $\delta \geq \frac{n}{2}$. Then G is Hamiltonian.

For D-cycles, a theorem of Nash-Williams (see [2]) establishes an upper bound, which an example of Veldman [8] shows is best possible:

Theorem A2. Let G be a c-connected graph $(c \geq 2)$ with $\delta > \frac{n+1}{3}$. Then G has a D-cycle.

Both these results give sufficiency conditions with respect to δ depending only on n. As long as the connectivity is high enough, it is irrelevant.

Before we prove results about dominating cycles, we need a lemma, which relies on the following two theorems. Bondy [2] gives Theorem B, which relates connectivity, minimum degree and what can lie off a longest cycle. A graph is n-path-connected if any two vertices are connected by a path of length at least n.

Theorem B. Let G be a c-connected graph such that the degree-sum of any $c+1$ independent vertices is at least $n+c(c-1)$, where $n \geq 3$, and let L be a longest cycle in G. Then $G-L$ contains no $(c-1)$-path-connected subgraph.

Theorem C, from Erdös and Gallai [6], relates number of edges and the length of the longest cycle.

Theorem C. Let G be a graph on n vertices with at least $\frac{1}{2}d(n-1)+1$ edges, where $d>1$. Then G contains a cycle of length at least $d+1$.

Lemma 1 follows directly from these theorems.

Lemma 1. Let G be a c-connected graph, $c \geq 3$, with $\delta \geq \frac{n}{c+1}+c-1$ and let L be a longest cycle in G. Then all subgraphs H in $G-L$ have less than $(c-2)(v(H)-1)+1$ edges.

Proof. Let H be a subgraph of $G-L$. From Theorem C, H is not $(c-1)$-path connected, which implies no cycles of length $2c-3$ or more since such a cycle is $(c-1)$-path-connected. Using Theorem D, H must have less than $\frac{1}{2}(2c-4)(v(H)-1)+1$ edges. □

2. Dominating cycles in graphs with small connectivity.

Our goal is to establish a sufficiency condition for the existence of dominating cycles. In order to establish a general pattern, we begin by proving sufficiency conditions with respect to δ for dominating cycles in graphs with small connectivity. Later we extrapolate this pattern to formulate a conjecture about the sufficiency condition.

Lemma 2. Let G be a connected graph with $n \geq 3$ and $\delta \geq \frac{n}{2}$. Then G contains a dominating cycle.

Proof. From Theorem A, G has a Hamiltonian cycle. A Hamilton cycle dominates. □

Lemma 3. Let G be a 2-connected graph with $n \geq 3$ and $\delta \geq \frac{n}{3}$. Then G contains a dominating cycle.

Proof. From Dirac [5], G has a cycle L of length at least $\frac{2n}{3}$. Since $\delta \geq \frac{n}{3}$, every vertex must have a neighbour on the cycle. □

Until this point, the situation for dominating cycles is essentially the same as it is for D-cycles; for $c \geq 3$, however, we see a radical difference.

Theorem 1. Let G be a 3-connected graph, with sufficiently large n and $\delta \geq \frac{n}{4}+2$. Then G has a dominating cycle.

Proof. Let L be a longest cycle in G. From Dirac [5], L has length at least 2δ. If L does not dominate then there exists some $v \in V(G)$ such that $V(H) \cap V(L) = \emptyset$, where $V(H) = N(v)$ and $v(H) \geq \delta$. By lemma 1,

$$\epsilon(G-L) < (c-2)(v(G-L)-1)+1 \leq \frac{n}{2}-4.$$

There must exist some $x, y \in V(H)$ such that $d_{G-L}(x) \leq 1$ and $d_{G-L}(y) \leq 1$ (see Figure 1).

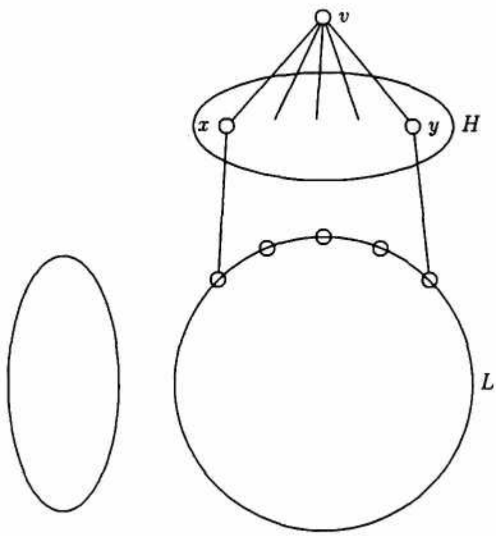

Figure 1.

Otherwise $d_{G-L}(u) \geq 2$ for all $u \in V(H)$ except for possibly some $x' \in V(H)$. By lemma 1, no $u \in V(H)$ can have neighbour $w \in V(H)$, since then $\epsilon(u+v+w) = 3$. Therefore each edge must account for one vertex degree and since $v(H) \geq \delta$

$$\sum_{\substack{u \in V(H) \\ u \neq x'}} d_{G-L}(x) < \epsilon(G-L) \quad \text{or} \quad 2(\frac{n}{4}+2-1) = \frac{n}{2}+2 > \frac{n}{2}-4.$$

Then $d_{L+x}(x) \geq \frac{n}{4}+1$ and $d_{L+y}(y) \geq \frac{n}{4}+1$. The neighbours of x and y on L must be at least four apart on L or we could form a longer cycle by including x,v,y and omitting the vertices on the cycle between the neighbours of x and y. Therefore $v(L) \geq 4(\frac{n}{4}+1) > n$. L must dominate. □

Theorem 2. Let G be a 4-connected graph, with sufficiently large n and $\delta \geq \frac{n}{5}+3$. Then G has a dominating cycle.

Proof. Let L be a longest cycle in G that dominates the most vertices. Again L has length at least 2δ. If L does not dominate then there exists some v and H as in theorem 1. By lemma 1,

$$\epsilon(G-L) < 2(\frac{3}{5}n-6-1)+1 = \frac{6}{5}n-13.$$

There must exist some $x,y \in V(H)$ such that $d_{G-L}(x) \leq 11$ and $d_{G-L}(y) \leq 11$. Otherwise $d_{G-L}(u) \geq 12$ for all $u \in V(H)$ except possibly for some $x' \in V(H)$. Since each edge can account for two vertex degrees and $v(H) \geq \delta$

$$\frac{1}{2}\sum_{\substack{u \in V(H) \\ u \neq x'}} d_{G-L}(x) < \epsilon(G-L) \quad \text{or} \quad \frac{1}{2}12(\frac{n}{5}+3-1) = \frac{6}{5}n+12 < \frac{6}{5}n-13.$$

The neighbours of x and y on L must be at least four apart or we could form a longer cycle. Consider any two neighbours of x and y on L that are four apart, l_1 and l_5, and the vertices between them on L, l_2, l_3 and l_4 (see Figure 2). All neighbours of l_3, other than l_2 and l_4, must not be on L or we could construct a cycle of equal length that dominates one more vertex by leaving l_2,l_3,l_4 off the cycle and including x,v,y. This would contradict the choice of L. Therefore if any neighbours of x and y on L are four apart, $v(L) \geq 4(\delta-11) = \frac{4}{5}n-32$ and $v(G-(L+H+v)) \geq \delta-2 = \frac{n}{5}+1$. But then G must have more than n vertices. If the neighbours of x and y on L are all at least five apart, then $v(L) \geq 5(\delta-11) = n-40$ and again we have more than n vertices. Therefore L must dominate. □

Theorem 3. Let G be a 5-connected graph, with sufficiently large n and $\delta \geq \frac{n}{6}+6$. Then G has a dominating cycle.

Proof. Let L be a longest cycle in G that dominates the most vertices. Again L has length at least 2δ. If L does not dominate then there exists some v and H as in theorem 1 and 2. By lemma 1,

$$\epsilon(G-L) < 3(\frac{2}{3}n-12-1)+1 = 2n-38.$$

Similar to theorem 2, there must exist some $x,y \in V(H)$ such that $d_{G-L}(x) \leq 23$ and

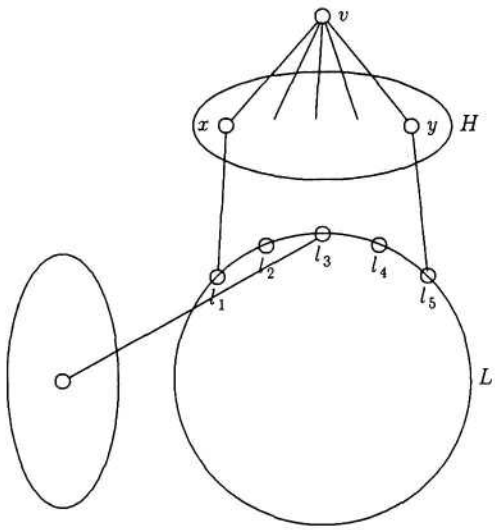

Figure 2.

$d_{G-L}(y) \leq 23$. The neighbours of x and y on L must be at least four apart or we could form a longer cycle. Consider any two sets of vertices of L that have two neighbours of x and y four apart, l_1,l_2,l_3,l_4,l_5 and $l_k,l_{k+1},l_{k+2},l_{k+3},l_{k+4}$ (see Figure 3).

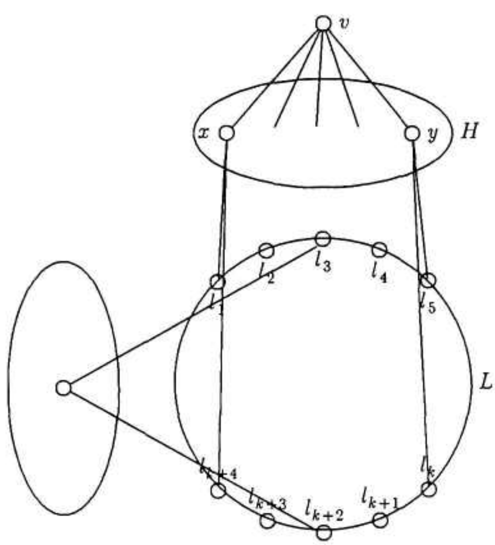

Figure 3.

Both l_3 and l_{k+2} must have all neighbours off the cycle, except for their immediate neighbours on the cycle. These neighbours must also be disjoint or we can form a longer cycle as indicated in Figure 3. Therefore if two or more sets of neighbours of x and y are four apart, $v(L) \geq 4(\delta-23) = \frac{4}{6}n-68$ and $v(G-(L+H+v)) \geq 2(\delta-2) = \frac{2}{6}n+8$. But then G as more than n vertices. If there is at most one set of neighbours of x and y that is four apart then $v(L) \geq 5(\delta-23)-1 = \frac{5}{6}n-86$. With this new estimate of the size of $v(L)$ by lemma 1,

$$\epsilon(G-L) < 3(\frac{n}{6}+86-1)+1 = \frac{n}{2}+256.$$

Similar to theorem 2, there must exist some $x,y \in V(H)$ such that $d_{G-L}(x) \leq 6$ and $d_{G-L}(y) \leq 6$. If there is at most one set of neighbours of x and y that is four apart then $v(L) \geq 5(\delta-6)-1 = \frac{5}{6}n-1$ and we again have more than n vertices. Therefore L dominates. □

3. The conjectured sufficiency condition

Even though we do not know the exact result for higher connectivity, the following example places a lower bound on the sufficiency condition. Let $c \geq 1$, $A \geq c$ and G consist of the following subgraphs:

$$X = K_c,$$
$$Y_i = K_A \vee z_i,\ i = 1,2, \cdots, c+1,$$

with extra edges from every vertex in X to every vertex in $Y_i - z_i$ (see Figure 4).

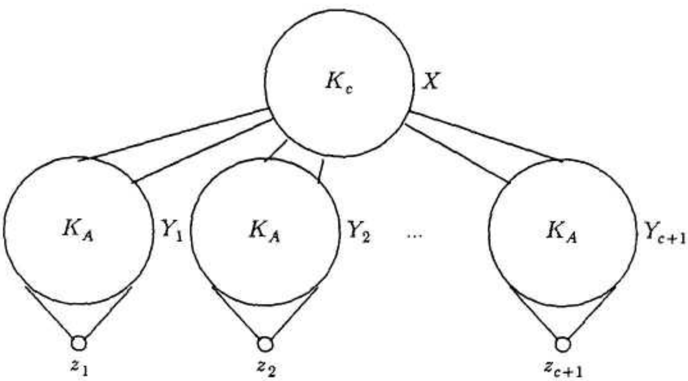

Figure 4.

G has connectivity c, $\delta = A = \frac{n}{c+1}-(1+\frac{c}{c+1})$ and no dominating cycle. For a dominating cycle to exist each Y_i must have at least one vertex on the cycle so each z_i will be adjacent to the cycle, but to include each Y_i we need $c+1$ vertices in X. G shows that the sufficiency condition is greater than $\frac{n}{c+1}-2$.

We conjecture the following sufficiency condition for dominating cycles in terms of δ and c:

Conjecture 1. Let G be a c-connected graph with $n \geq 3$ and $\delta \geq \frac{n}{c+1}+k_c$, where k_c is a constant depending only on c. Then G has a dominating cycle.

Conjecture 1 is the best possible by the previous example so it may allow values for δ that are too low to guarantee a dominating cycle. We are much more certain that the sufficiency condition for δ is not a constant, like $\frac{n}{6}$, as it is for Hamilton cycles and D-cycles. It may be more reasonable to try to find a sufficiency conditions, in terms of δ, for each c that are less than $\frac{n}{6}$ and decrease as c increases.

One hope in proving such a conjecture is to show that when there is a dominating cycle, some longest cycle dominates as we did in theorems 1,2 and 3. However, the following example shows that longest cycles are not necessarily dominating although dominating cycles exist. Given $c \geq 6$ and $m \geq 6$ we construct such a graph G with $\delta = \frac{n+4}{6}$ on $n=6m+2$ vertices. Let G consist of the following subgraphs:

$$H = v \vee H'$$

where v is a vertex and H' is a $\overline{K}_m$, and

$$J = K_m \vee \bigcup_{i=1}^{m} Y_i$$

where $Y_i = K_4$, with extra edges from every vertex in H' to every vertex in the K_m (see Figure 5).

Then G has connectivity c and $\delta = m+1 = \frac{n+4}{6}$. A longest cycle in G has all of the vertices of each Y_i, $i = 1,2, \cdots, m$, and also K_m; to include H would add 3 vertices, but would also remove a Y_i from the cycle thereby subtracting more than 3 vertices. No longest cycle is dominating, but a dominating cycle exists. For $\delta \geq \frac{n}{c+1}+k_c$, sufficiently large n, and $c \geq 6$ such examples exist so for higher connectivity we cannot prove conjecture 1 by showing that it implies a longest cycle dominates. Nevertheless, we expect that the conjecture holds, and these examples simply show that our *longest cycle* techniques cannot generalize.

Acknowledgements

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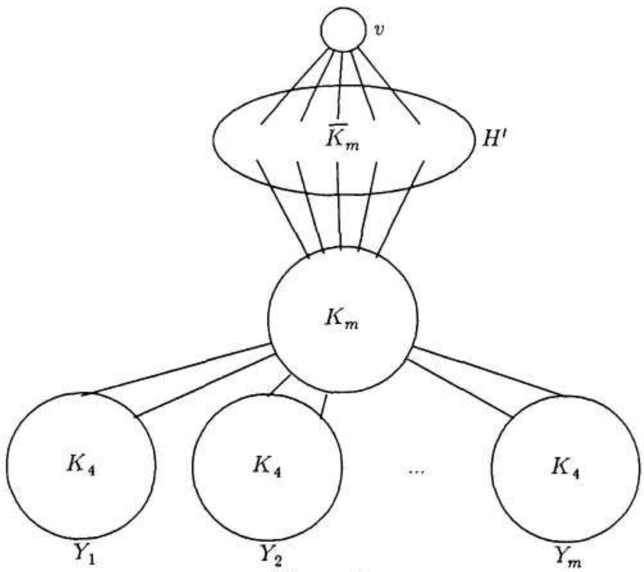

Figure 5.

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Note added (July 1985):

Bondy and Fan (University of Waterloo), and Fraisse (Université de Paris Sud) recently communicated to us different proofs of conjecture 1. We summarize Fraisse's proof here. Veldman [9] defines a D_λ cycle to be a cycle C for which G-C contains no connected component with λ or more vertices. Further define α_λ to be the maximum number of remote subgraphs of order λ (two subgraphs are *remote* if no edge connects a vertex in one to a vertex in the other). Veldman [9, thm 2] proves that if $\alpha_\lambda \leq c$, then G is D_λ cyclic when $c \geq 2$.

In general, a D_λ cycle need not be dominating, but if $\lambda \leq \delta+1$ such a cycle *is* dominating. Now set $\lambda = \frac{n-c}{c+1}$ and set $\delta \geq \lambda+c$. Compute α_λ. If $\alpha_\lambda \leq c$, then Veldman's theorem assures us that there is a D_λ cycle, which (since $\lambda < \delta$) is dominating. If on the other hand, $\alpha_\lambda > c$, there are at least $c+1$ remote subgraphs each with λ vertices. All are disjoint, and this accounts for $n-c$ vertices in total. The graph is c-connected, and hence the remaining c vertices are connected to each of the remote subgraphs. However, consider a vertex in one of the remote subgraphs. It has at most $\lambda-1$ neighbours in the remote subgraph, no neighbours in any other remote subgraph, and at most c neighbours among the connecting vertices. But then its degree is smaller than δ, which is a contradiction. This proves conjecture 1.

Bondy and Fan prove a more general theorem which has this result as a corollary.

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