ON THE EXISTENCE OF TWO NON-NEIGHBORING SUBGRAPHS IN A GRAPH

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Does there exist a function $f(r, n)$ such that each graph G with $\chi(G) \geqq f(r, n)$ contains either a complete subgraph of order r or else two non-neighboring n-chromatic subgraphs? It is known that $f(r, 2)$ exists and we establish the existence of $f(r, 3)$. We also give some interesting results about graphs which do not contain two independent edges.

1. Introduction

Two subgraphs G_1, G_2 of a graph G are called non-neighboring if there is no edge $v_1v_2 \in E(G)$ with $v_1 \in G_1$ and $v_2 \in G_2$. In general, an arbitrary graph may not contain two non-neighboring subgraphs at all, for example the complete graph. In this paper we raise the following question: Is there a minimal integer $f(r, n)$ such that each graph G with $\chi(G) \geqq f(r, n)$ and which does not contain a complete subgraph of order r must contain two non-neighboring n-chromatic subgraphs? An upper bound for $f(r, 2)$ follows from a result of S. Wagon [2]. Here we show that it is sufficient to prove the existence of $f(r, n)$ for $r \leqq n$. More precisely, for a fixed n, an upper bound for $f(r, n)$, $r > n$, is given in terms of $f(r, n)$, $r \leqq n$. The proof is based on the same idea of S. Wagon. From $f(3, 3) \leqq 8$ we deduce an upper bound for $f(r, 3)$. Next we investigate graphs which do not contain $2K_2$ as an induced subgraph. We say that the two edges v_1v_2, u_1u_2 of the graph G are independent if the subgraph induced by v_1, v_2, u_1, u_2 is $2K_2$, i.e., the complement of a chordless 4-cycle. We prove that a vertex-critical 4-chromatic graph G which does not contain two independent edges has order $|G| \leqq 13$. We also give a lower bound for the maximum degree of a graph without two independent edges.

2. Notation

We consider graphs $G = (V(G), E(G))$ which are finite, loopless and have no multiple edges. The neighborhood $N(v)$ of a vertex $v \in G$ is the set of vertices adjacent to v. We put $N^*(v) = N(v) \cup \{v\}$. For $W \subseteq V(G)$, we denote $N(W) = \cup \{N(v) : v \in W\}$ and $N^*(W) = N(W) \cup W$. If G_1 is a subgraph of G then $N(G_1)$, $N^*(G_1)$ respectively denote $N(V(G_1))$, $N^*(V(G_1))$. Two subgraphs G_1, G_2 are non-neighboring if $V(G_1) \cap \cap N^*(G_2) = \emptyset$. A subset $W \subseteq V(G)$ is called a dominating set if $N^*(W) = V(G)$.

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3. The functions $f(r, 2)$ and $f(r, 3)$

S. Wagon [2] proved that if G contains neither a complete subgraph of order r nor two independent edges then $\chi(G) \leqq \binom{r}{2}$. It follows that $f(r, 2) \leqq \binom{r}{2}+1$. The slightly stronger result $f(r+1, 2) \leqq f(r, 2)+r$ is implicit in [2]. It is trivial that $f(2, 2)=2$ and the pentagon C_5 shows that $f(3, 2)=4$. The 5-wheel C_5+K_1 shows that $f(4, 2) \geqq 5$ and from Wagon's Theorem we have $f(4, 2) \leqq 7$. Recently P. Hajnal proved that $f(4,2) \leqq 6$. Finally, Nagy and Szentmiklóssy proved $f(4,2)=5$.

Theorem 1. *For* $r > n$,

$$f(r, n) \leqq 1+(n-1)\binom{r-1}{n}+\sum_{j=1}^{n-1}(f(j+1, n)-1)\binom{r-1}{j}.$$

Proof. Let G be a graph which does not contain two non-neighboring n-chromatic subgraphs. Let K be a complete subgraph of maximum order in G and assume that $|K|=k \geqq n$. For each $1 \leqq j \leqq n$, let $S_j^{(i)}$, $1 \leqq i \leqq \binom{k}{j}$, denote the j-subsets of $V(K)$. Put

$$X_n^{(i)} = \{v \colon N(v) \cap S_n^{(i)} = \emptyset\} \qquad 1 \leqq i \leqq \binom{k}{n},$$

$$Y_j^{(i)} = \{v \colon N(v) \cap V(K) = V(K)-S_j^{(i)}\} \quad 1 \leqq j < n, \quad 1 \leqq i \leqq \binom{k}{j}.$$

We have $\chi(X_n^{(i)}) \leqq n-1$ since otherwise, $S_n^{(i)}$ would be non-neighboring to an n-chromatic subgraph of $X_n^{(i)}$. Also $\chi(Y_j^{(i)}) \leqq f(j+1, n)-1$ since $Y_j^{(i)}$ does not contain a complete subgraph of order $j+1$. The union of the $X_n^{(i)}$ and $Y_j^{(i)}$ is $V(G)$. Therefore

$$\chi(G) \leqq (n-1)\binom{k}{n}+\sum_{j=1}^{n-1}(f(j+1, n)-1)\binom{k}{j}$$

which implies the required result. ∎

Theorem 2. $f(3,3) \leqq 8$.

Proof. Let G be a triangle-free graph which does not contain two non-neighboring odd circuits. Let $C=v_0 v_1 \dots v_{2k}$ be an odd circuit of minimum lenght in G. We describe a proper 7-coloring c of G as follows. Let $c(v_0)=1$ and $c(v_i)=2$ (resp. 3) if i is odd (resp. even) and $1 \leqq i \leqq 2k$. Further we let $c(x)=2$ (resp. 3) if $x \in N(v_i)$ for i even (resp. odd) and $2 \leqq i \leqq 2k-1$. Otherwise, for $x \in N(C)$ we let

$$c(x) = \begin{cases} 1 & x \in N(v_1) \\ 4 & x \in N(v_0) \\ 5 & x \in N(v_{2k}). \end{cases}$$

Since $G-N^*(C)$ does not contain an odd circuit, we need at most two more colors 6 and 7 to extend c to all of $V(G)$. This shows that $\chi(G) \leqq 7$ and, therefore, $f(3,3) \leqq 8$. ∎

It is easy to check that the triangle-free 5-chromatic graph described by Mycielski [1] does not contain two non-neighboring odd circuits. This shows that $f(3,3) \geqq 6$.

From Theorems 1 and 2 we get the following polynomial upper bound for $f(r, 3)$.

Corollary 3. $f(r, 3) \leqq 2\binom{r-1}{3} + 7\binom{r-1}{2} + r \quad (r > 3)$. ∎

4. Graphs without two independent edges

In this section we prove some more results about graphs without two independent edges. We start by a result about 4-critical (i.e vertex-critical 4-chromatic) graphs without two such edges. Examples of these graphs are K_4 and the 5-wheel $C_5 + K_1$. We shall encounter more in what follows. It is somewhat a surprising fact that these graphs cannot have a large order, specially if we know that this is not the case for higher chromatic numbers.

Theorem 4. *If G is a 4-critical graph without two independent edges then* $|G| \leqq 13$.

Proof. We may assume that G is not (and therefore does not contain) K_4. Let $v_1 v_2 v_3$ be a triangle in G. We have two cases.

Case 1. There is a vertex $v \in G$ adjacent to none of v_1, v_2, v_3. Since G contains no two independent edges, then each vertex $u \in N(v)$ must be adjacent to exactly two of v_1, v_2, v_3. Let c be a proper 3-coloring of $G - v$. There must exist three vertices $u_1, u_2, u_3 \in N(v)$ such that $c(u_i) = i_i (i = 1, 2, 3)$. Suppose the vertices v_1, v_2, v_3 were so labelled that $c(v_i) = i$. Thus u is adjacent to v_j iff $i \neq j$. However, G contains no more vertices since, so far, it is 4-critical. G could contain none, one or two more edges connecting some of u_1, u_2, u_3. These graphs are shown in Figure 1.

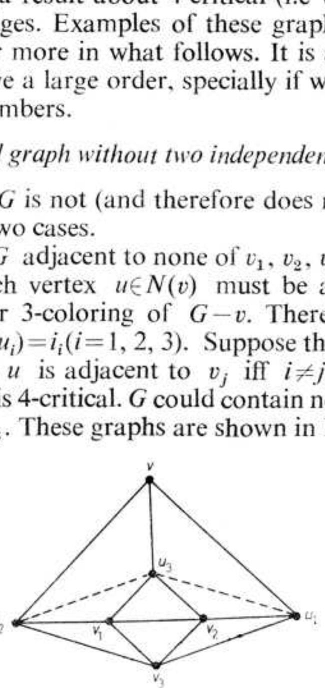

Fig. 1

Case 2. $N^*(v_1, v_2, v_3) = V(G)$.
We let

$$A_i = N(v_i) - \bigcup_{j \neq i} N(V_j), \quad B_i = \bigcap_{j \neq i} N(v_j) - \{v_i\}, \quad 1 \leqq i \leqq 3,$$

$$A = \bigcup_{i=1}^{3} A_i \quad \text{and} \quad B = \bigcup_{i=1}^{3} B_i.$$

We note that each B_i is independent since G contains no K_4. Also, each A_i is independent since if, for example, there were two adjacent vertices $a, a' \in A_1$ then the two edges aa', v_2v_3 would be independent. Therefore both A and B are nonempty. We pick an edge xy with $x \in A_i, y \in B_j$ and $i \neq j$ (if no such edge exists then G is 3-chromatic). For convenience, we assume that $y \in B_3$ and $x \in A_2$. We choose a maximal uniquely 3-colorable subgraph in $G-x$ as follows. Assign color i to v_i ($i=1, 2, 3$). At each subsequent step a vertex v is assigned color j whenever it is adjacent to two vertices which were previously assigned the two other colors distinct from j. We continue in this way until we cannot proceed further. Denote by W the set of vertices colored in this way and by $c(v)$ the color assigned to $v \in W$. Thus, for example, $c(v)=i$ for each $v \in B_i$. Suppose $c(w) \neq 1$ for each vertex $w \in W$ adjacent to x. Then by putting $c(x)=1$ and

$$c(v) = \begin{cases} 1 & v \in A_2 - W \\ 2 & v \in A_3 - W \\ 3 & v \in A_1 - W \end{cases}$$

we get a proper 3-coloring of G which is a contradiction. Therefore, there is a vertex $w_1 \in W$ adjacent to x with $c(w_1)=1$. We prove that there is a path $w_1 w_2 \dots w_t$ such that:

(i) $c(w_i) \equiv i \pmod 3$

(ii) for $i < t$, $w_i \in A_j$ where $i+2 \equiv j \pmod 3$ and $w_t \in B_l$ where $t \equiv l \pmod 3$.

This is true if $w_1 \in B_1$. Suppose not, then necessarily $w_1 \in A_3$ and it was assigned color 1 due to its adjacency to a vertex w_2 (previously) colored with color 2. Either $w_2 \in B_2$ or $w_2 \in A_1$ and we can find w_3 with the required properties. Continuing in this way we, eventually, arrive at $w_t \in B$. Let us assume further that this path is of minimum length. Clearly the vertices $v_1, v_2, v_3, x, y, w_1, \dots, w_t$ span a 4-chromatic graph so that they must be all of $V(G)$. Thus we have to prove that $t \leqq 8$. Assume, on the contrary, that $t \geqq 9$. Consider the two edges xw_1 and w_7w_8. There is no edge w_1w_7 since $c(w_1)=c(w_7)=1$. Also, $xw_7, w_1w_8 \notin E(G)$ since, otherwise, we could have chosen a path of smaller length. Therefore $xw_8 \in E(G)$. Now consider the two edges xw_8 and w_4v_3. If either $xw_4, w_4w_8 \in E(G)$ then we get a path of smaller length. Also none of x, w_8 is adjacent to v_3. This is a contradiction since xw_8, w_4v_3 cannot be independent. This completes the proof that $|G| \leqq 13$. ∎

To show that 13 in Theorem 4 is best possible, we give a graph G with $|G|=13$. This is shown in Figure 2.

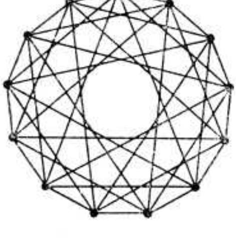

Fig. 2

In contrast to Theorem 4, we describe a 5-critical graph without two independent edges which has $4n+5$ vertices for arbitrary n. The vertices of this graph are $x_1, x_2, x_3, x_4, y_0, y_1, \dots, y_{4n}$. The edges are

$$\begin{array}{ll} x_i x_j & (i \neq j), \\ y_i y_{i+1} & (0 \leqq i \leqq 4n-1), \\ y_i x_j & i-j \equiv 2, 3 \pmod 4, \\ y_i y_j & i > j \quad \text{and} \quad i-j \equiv 2, 3 \pmod 4, \\ y_0 x_3 & y_{4n} x_4 . \end{array}$$

Our next result describes dominating sets of connected graphs without two independent edges. Here, of course, the connectedness is equivalent to having no isolated vertices.

Theorem 5. *Let G be a connected graph without two independent edges. Then G has a dominating set whose induced subgraph is either a complete subgraph or a path on 3 vertices.*

Proof. Let $v_1 v_2 \in E(G)$. Denote $X = V(G) - N^*(v_1, v_2)$ and $Y = N^*(v_1, v_2) - \{v_1, v_2\}$. We may assume $X \neq \emptyset$ since otherwise $\{v_1, v_2\}$ is a dominating set. The set X is independent but each $x \in X$ is adjacent to at least one $y \in Y$. We choose vertices $y_1, \dots, y_r \in Y$ with r minimum and satisfying:

(i) $X \subseteqq \bigcup_{i=1}^{r} N(y_i)$,

(ii) for each i, $N(y_i) \cap X$ is maximal that is not properly contained in $N(y) \cap X$ for any $y \in Y$.

If $r=1$ then $\{v_1, v_2, y_1\}$ is a dominating set with the required property. Let us assume that $r \geqq 2$. If $y_i \neq y_j$ then we can find $x, x' \in X$ such that $y_i x$, $y_j x' \in E(G)$ but $y_i x'$, $y_j x \notin E(G)$. Therefore $y_i y_j \in E(G)$ for all $i \neq j$. Obviously $\{v_1, v_2, y_1, \dots, y_r\}$ is a dominating set. So we need only to prove that for $i = 1, 2$, either v_i is adjacent to all of $y_1, \dots, y_r$ or else $N(v_i) \subseteqq N(y_1, \dots, y_r)$. Suppose on the contrary that, for example, $v_1 y_1 \notin E(G)$ and there is a vertex $v \in N(v_1)$ adjacent to none of $y_1, \dots, y_r$. Let $x \in N(y_1) \cap X$. Since the two edges $v_1 v$, $y_1 x$ are not independent, then we must have $xv \in E(G)$. Therefore $N(y_1) \cap X \subseteqq N(v) \cap X$ and by (ii) above equality holds. Choose two vertices $x_1, x_2 \in X$ with $x_1 y_1, x_2 y_2 \in E(G)$ and $x_1 y_2, x_2 y_1 \notin E(G)$. The two edges $x_1 v$, $x_2 y_2$ are then independent. This is a contradiction and our theorem is proved. ∎

Corollary 1. *If G is a connected graph of order n and without two independent edges then its maximum degree* $\Delta(G) \geqq \min\left\{2\sqrt{n} - 2, \frac{1}{3}(n+1)\right\}$.

Proof. If G has the vertices of a complete subgraph of order r as a dominating set then this complete subgraph contains a vertex x with degree

$$d(x, G) \geqq \frac{1}{r}(n-r) + r - 1 \geqq 2\sqrt{n} - 2.$$

If the dominating set is a path $v_1 v_2 v_3$ then for some i

$$d(v_i, G) \geqq \frac{1}{3}(n+1). \quad \blacksquare$$

For each n, we can construct a graph G with $|G|=n$, $\Delta(G)=\lceil 2\sqrt{n}-2\rceil$ ($\lceil x\rceil$ is the smallest integer $\geqq x$) and no two independent edges as follows. Choose two positive integers r, s with $rs \geqq n$ and $r+s$ is minimum. Starting from the vertices $v_1, \ldots, v_r$ of the complete graph K_r, we add $n-r$ more vertices $u_1, \ldots, u_{n-r}$ each joined to at least one v_i in such a way that no v_i is joined to more than $s-1$ vertices u_j. Clearly for large n these are the only extremal graphs to Corollary 1. In particular, for sufficiently large n, $\lceil 2\sqrt{n}-2\rceil$ is the smallest possible maximum degree. However, even for small values of n, this is not far from being true.

Corollary 2. *All connected graphs G on n vertices and without two independent edges satisfy $\Delta(G) \geqq 2\sqrt{n}-2$ except the three graphs shown in Figure 3.*

Proof. Assume $\Delta(G) < 2\sqrt{n}-2$. Then, since $\Delta(G)$ is an integer, we must have $\left\lceil \frac{1}{3}(n+1) \right\rceil < 2\sqrt{n}-2$. This is true only for $n=5, 7, 8, 10, 11, 13, 14, 17$. However, keeping in mind that G must have a path of three vertices as a dominating set, we can check each case to see that no such graph exists except when $n=5, 7, 10$ where there is a unique graph in each case. $\blacksquare$

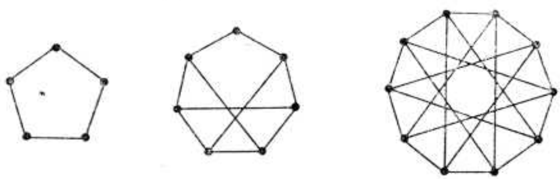

Fig. 3

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