PROBLEMS AND RESULTS IN COMBINATORIAL GEOMETRY[a]

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I have published during my very long life many papers with similar titles. In this note, I report on progress made on some of the old problems and discuss also some new ones, and I repeat some old problems which seem to have been forgotten by everybody (sometimes including myself). First, a very incomplete list is given of some papers dealing with these and related questions.

I have published three survey papers on this subject [1–3].

I want to call attention to the excellent book of Hadwiger and Debrunner which originally appeared in German and French. This book was translated by Klee who also added much original material and brought the book up-to-date (up to about 1964) [4]. This book contains a great deal of interesting material and can be used as a textbook to learn the subject, but contains few unsolved problems. (Unfortunately, it is out of print.)

A very fruitful source of problems and results is the much expanded 1981 edition of W. Moser's collection, "Research Problems in Discrete Geometry" (McGill University). The nucleus of this collection was the old problems of the late Leo Moser.

In *Elemente der Mathematik* there is a section entitled "Ungelöste Probleme" which was edited by Hadwiger and in the *American Mathematical Monthly* there is a similar section edited by Guy; both often contain geometrical problems. See also the papers by Fejes-Toth, Grünbaum, and Klee that are quoted in [1].

G. Purdy and I plan to write a book on various problems in combinatorial geometry. We hope the book will appear in this decade.

1

Let there be given n distinct points $x_1, \ldots, x_n$ in k-dimensional Euclidean space E_k. Denote by $d(x_i, x_j)$ the distance between x_i and x_j. $D_k(x_1, \ldots, x_n)$ denotes the number of distinct distances among the x_i's. Put

$$f_k(n) = \min_{x_1, \ldots, x_n} D_k(x_1, \ldots, x_n).$$

Denote further by $P_k(n)$ the largest integer for which there are $P_k(n)$ pairs x_i and x_j for which $d(x_i, x_j) = 1$. Both these functions have been extensively studied in [1] (with many references).

The best results until recently were

$$P_2(n) = o(n^{3/2}), \qquad f_2(n) > cn^{2/3}. \tag{1}$$

[a] Dedicated to the memory of my friend Hugo Hadwiger.

The first result in (1) is due to Szemerédi, the second to L. Moser. Recently, J. Beck and J. Spencer proved

$$P_2(n) < n^{3/2-\varepsilon} \qquad \text{for some } c > 0 \tag{2}$$

and Fan Chung proved

$$f_2(n) > cn^{5/7}. \tag{3}$$

I still believe that my old conjectures

$$f_2(n) > cn/(\log n)^{1/2}, \qquad P_2(n) < n^{1+c/\log\log n} \tag{4}$$

are true and I offer \$500 for a proof or disproof. I offer \$250 for $P_2(n) < n^{1+\varepsilon}$.

Observe that L. Moser's proof in fact gives that there is a point x_i so that the number of distinct distances $d(x_i, x_j)$, $i \leq j \leq n$, $j \neq i$, is greater than $cn^{2/3}$. The proof of Fan Chung does not seem to give $cn^{5/7}$, but perhaps it is true. In fact, perhaps there always is a point x_1 so that the number of distinct distances $d(x_1, x_i)$, $2 \leq i \leq n$, is greater than $cn/(\log n)^{1/2}$. This and several other conjectures are discussed in [1].

For $k \geq 4$, $P_k(n)$ is much easier to handle than it is for $k = 2$.

Let $x_1, \ldots, x_n$ be n points on the surface of a two-dimensional sphere. Denote by $P'_2(n)$ the largest integer for which there are $P'_2(n)$ pairs x_i, x_j for which all the distances $d(x_i, x_j)$ are the same. I could not prove that

$$\lim_{n\to\infty} P'_2(n)/n = \infty.$$

Ulam recently asked me the following question: Let $x_1, \ldots, x_n$ be n points in the plane. Does one get interesting combinatorial and geometric questions if one modifies the metric and asks how often we can have $d(x_i, x_j) = 1$? E.g., he asked: What if we define the distance of two points as the sum of the absolute values of the differences of their coordinates? In this case I proved that if $n > 4$, $n \equiv 0 \pmod 4$, then $P_2(n) = (n^2 + n)/4$. I hope to return to these questions later.

Reference for Section I: [5].

II

Let there be given n points in the plane not all on a line. Join every two of them. Sylvester conjectured and Gallai proved in 1933 that there always is a line that passes through precisely two of these points. This problem and various extensions and generalizations are of course discussed in [1]. Motzkin conjectured that for $n > 13$, the number of ordinary lines (i.e., the number of lines going through precisely two of the points) is $\geq n/2$. Motzkin observed that for every even n, this conjecture—if true—is best possible. Hansen recently proved this conjecture; his proof, which is not yet published, is quite complicated [6].

Denote by $L_1, \ldots, L_m$ the lines determined by our points $x_1, \ldots, x_n$. Denote by $|L_i|$ the number of points on L_i. Put $|L_i| = y_i$, $y_1 \geq \cdots \geq y_m$. It easily follows from Gallai's theorem and also from a theorem of de Bruijn and myself that if $m > 1$, then $m \geq n$. Grünbaum asked, What are the possible values of m? I showed that for

$m > cn^{3/2}$ all possible values of m can occur except $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$. The analogous problems in higher dimensions can probably be settled without great difficulty (lines are replaced by hyperplanes).

The following problem is probably quite difficult: Denote by $F(n)$ the number of possible choices of the set $\{y_1, \ldots, y_m\}$. We must of course have

$$\sum_{i=1}^{m} \binom{y_i}{2} = \binom{n}{2}. \tag{5}$$

(5) is by no means sufficient for the existence of a set $x_1, \ldots, x_n$ for which $|L_i| = y_i$ and perhaps it is not reasonable to expect a good necessary and sufficient condition. I conjectured several years ago that

$$F(n) < \exp cn^{1/2}. \tag{6}$$

It is easy to see that (6) if true is best possible, but I do not believe that to decide (6) will be easy and I offer $250 for a proof or disproof of (6).

Denote by $h_k(x_1, \ldots, x_n)$ the number of indices i for which $|L_i| = k$. What is the possible range of values of $h_k(x_1, \ldots, x_n)$? As far as I know, this question has not yet been investigated.

Gallai's theorem can be stated as follows: Unless all the x_i are on a line, $h_2(x_1, \ldots, x_n) \geq 1$. Hansen's theorem states that for $n > 13$, $h_2(x_1, \ldots, x_n) \geq n/2$. Put

$$\max_{x_1, \ldots, x_n} h_k(x_1, \ldots, x_n) = t_k(n).$$

Trivially $t_2(n) = \binom{n}{2}$. $t_3(n)$ was first studied by Sylvester and was investigated in a very nice paper by Burr, Grünbaum, and Sloane [7].

Croft and I observed that for every k and $n > n_0(k)$, $t_k(n) > c(k)n^2$. We could not determine the largest possible value of $c(k)$. Put

$$\lim_{n \to \infty} t_k(n)/n^2 = c_k.$$

Trivially $c_k \leq 1/k(k-1)$ and for $k = 2$ and 3, equality is possible. Croft and I conjectured that

$$\lim_{k \to \infty} k^2 c_k = 0, \tag{7}$$

but we could not even prove $c_k < 1/k(k-1)$ for $k > 3$. (Recently, Szemerédi and Trotter proved $c_k < (\log k)^c/k^3$, where c is an absolute constant.) Put

$$H(n) = \max_{x_1, \ldots, x_n} \sum_{k \geq \sqrt{n}} h_k(x_1, \ldots, x_n).$$

In other words, $H(n)$ is the largest number of lines that contain at least $\sqrt{n}$ points. I conjectured that $H(n) < cn^{1/2}$, but could not even prove $H(n) = o(n)$. Recently, Szemerédi and Trotter proved $H(n) < cn^{1/2}$. This result has not yet been published.

I thought that perhaps $H(n^2) = 2n + 2$, but Szemerédi informed me that recently someone showed $H(n) > 3n^{1/2}$.

Purdy and I conjectured that

$$\max \sum_{i=1}^{n} |L_i| < cn^{4/3}. \tag{8}$$

The lattice points show that (8) if true is best possible. It is possible that the method of Szemerédi and Trotter will give (8). More generally, perhaps the number of indices i for which $|L_i| > n^{1/2}2^{-u}$ is less than $c2^{2u}n^{1/2}$ and if true the lattice points show that it is best possible. Szemerédi and Trotter have proven

$$\max \sum_{i=1}^{n} |L_i| < c_1 n^{4/3}(\log n)^{c_2}.$$

Croft and I conjectured that to every $\varepsilon > 0$ there is a $k_0 = k_0(\varepsilon)$ so that if $|L_i| < \varepsilon n$ for all i, then

$$\sum_{|L_i| > k_0} \binom{L_i}{2} < \varepsilon \binom{n}{2}.$$

In other words, the number of pairs of points situated on "large lines" is small.

Assume that $x_1, \ldots, x_n$ is such that $|L_i| \leq k$; i.e., no line contains more than k points. Denote by $T_k(n)$ the maximum number of lines that contain k points. I conjectured that for $k > 3$

$$T_k(n) = o(n^2). \tag{9}$$

For $k = 3$, Sylvester and Burr, Grünbaum, and Sloane [7] proved that $T_3(n) = (1 + o(1))(n^2/c)$, but the exact value of $T_3(n)$ is not yet known.

Grünbaum proved

$$T_k(n) > cn^{1+1/k};$$

perhaps the exponent $1 + 1/k$ is best possible.

Purdy observed that

$$\frac{1}{m} \sum_{i=1}^{m} |L_i| < 3$$

and that 3 is best possible.

Additional references for Section II: [8, 9].

III

Hadwiger and Nelson posed the following problem: Let G_k be a graph whose vertices are the points of E_k (the k-dimensional Euclidean space). Two points are joined if their distance is 1. Determine or estimate the chromatic number $h(k)$ of G_k. It is known that $4 \leq h(2) \leq 7$. I am almost sure that $h(2) > 4$. In this connection, L. Moser asked the following interesting question: Let R be large and S a measurable set in the circle of radius R so that no two points of S have distance 1. Denote by $m(S)$ the measure of S. Determine

$$\lim_{R\to\infty} \max m(S)/R^2. \tag{10}$$

It seems very likely that the limit in (10) is less than $\frac{1}{4}$.

I asked the following question: Let S be a subset of the plane. Join two points of S if their distance is 1. This gives the graph $G(S)$. Assume that the girth (shortest circuit) of $G(S)$ is k. Can its chromatic number be greater than 3? Wormald proved that such a graph exists for $k \leq 5$. The problem is open for $k > 5$. Wormald suggested that his method may work for $k = 6$, but probably a new idea would be needed for $k > 6$.

A related (perhaps identical) question is: Does $G(S)$ have a subgraph that has girth k and chromatic number 4?

Let $0 < r_1 < \cdots < r_k$. $G_2(r_1, \ldots, r_k)$ is the graph whose vertices are the points of the plane and two points are joined if their distance is one of the numbers $r_1, \ldots, r_k$. Put ($h(G)$ is the chromatic number of G)

$$h_r(2) = \max_{r_1, \ldots, r_k} h(G_2(r_1, \ldots, r_k)).$$

It is easy to see that $\lim_{r\to\infty} h_r(2)/r = \infty$, but I do not know if $h_r(2)$ grows polynomially.

Larman and Rogers proved

$$ck^2 < h(k) < (3 + o(1))^k.$$

I conjectured that $h(k) > (1 + c)^k$. This conjecture was recently proved by Frankl and Wilson. The value of $\lim_{k\to\infty} h(k)^{1/k}$ is not known and in fact it is not even known if the limit exists.

V. T. Sós and I proved that if there are $n + 1$ triples in a set S of n elements, then there are always two of them whose intersection is a singleton. The proof is simple. We conjectured that if $|S| = n$, $n > n_0(k)$ and $A_i \subset S$, $|A_i| = k$, $1 \leq i \leq t_k$, $t_k > \binom{n-2}{k-2}$ is a family of distinct subsets of S, then there are always two A's whose intersection is a singleton. It was immediate that if true, this conjecture is best possible.

Katona proved the conjecture for $k = 4$ and Frankl proved it in the general case.

I further conjectured: Let $|S| = n$, $0 < \eta < \frac{1}{4}$, $A_i \subset S$, $1 \leq i \leq T_{n,\eta}$. Assume that for some r,

$$\eta n < r < (\tfrac{1}{2} - \eta)n, \qquad (A_i \cap A_j) \neq r \text{ for every } 1 \leq i < j \leq T_{n,\eta}.$$

Then

$$T_{n,\eta} < (2 - \varepsilon)^n, \qquad \varepsilon = \varepsilon(\eta). \tag{11}$$

It is easy to see that (11) implies $h(k) < (1 + c)^k$. (11) is still open; the proof of Frankl and Wilson did not use (11). [June 25, 1982—I just had a letter from Frankl: He and Füredi have proved (11).]

In a recent paper Simonovits and I investigated $h^*(k)$, the so-called essential chromatic number of E_k. $h^*(k)$ is the largest integer for which there is a finite $x_1, \ldots, x_n$ in E_k so that if we omit $o(n^2)$ edges of G_k in all possible ways, we are always left with a graph of chromatic number $\geq h^*(k)$. We prove that $h^*(4) = 2$ and conjecture that $h^*(k)$ tends to infinity exponentially. In fact, we only proved $h^*(k) \geq k - 2$. Several further problems (which I think are interesting and challenging) are discussed in our paper.

A well-known theorem of de Bruijn and myself states that every k-chromatic graph contains a finite subgraph of chromatic number k. Thus all the problems considered in this section are problems about finite sets and finite graphs.

Perhaps the following modification of the Hadwiger–Nelson problem is of some interest: Join two points in E_2 if their distance is between α and β, $\alpha < 1 < \beta$. Determine or estimate the chromatic number $h(2, \alpha, \beta)$ of this graph.

References for Section III: [10–16].

IV

An old problem of Heilbronn states as follows: Let $z_1, \ldots, z_n$ be n points in the unit circle. Denote by $A_k(z_1, \ldots, z_n)$ the smallest area of all polygons $\{z_{i_1}, \ldots, z_{i_k}\}$, $1 \leq i_1 < \cdots < i_k \leq n$. Put

$$g_k(n) = \max_{z_1, \ldots, z_n} A_k(z_1, \ldots, z_n).$$

Heilbronn asked for the determination or estimation of $g_3(n)$. He of course observed that trivially $g_3(n) < c/n$ and suspected that the order of magnitude of $g_3(n)$ is $1/n^2$. I observed that indeed $g_3(n) > c/n^2$. The first nontrivial result was due to K. F. Roth who proved $g_3(n) < 1/n(\log \log n)^{1/2}$. This was improved by W. Schmidt to $c/n(\log n)^{1/2}$ and later by Roth to $1/n^{1+c_1}$.

In a recent paper, Komlós, Pintz, and Szemerédi [17] improved the value of c_1, but their really surprising result was their proof of

$$g_3(n) > \frac{c \log n}{n^2}. \tag{12}$$

The proof of **(12)** uses a novel combination of combinatorial and probabilistic arguments, which will have many further applications. Szemerédi believes that **(12)** perhaps is best possible.

As far as I know, the first nontrivial results for $k > 3$ are due to Schmidt. He proved

$$g_k(n) > \frac{c_k}{n^{1+1/k}} \tag{13}$$

and observed that the proof of

$$g_4(n) = o\left(\frac{1}{n}\right) \tag{14}$$

presents considerable difficulties. As far as I know **(14)** is still open and seems to be a fundamental problem.

Szemerédi and I posed the following problem: Denote by $D(z_1, \ldots, z_n)$ the smallest distance between two of our z's and $\alpha(z_1, \ldots, z_n)$ is the smallest angle determined by three of our points (if three z's are on a line, then $\alpha(z_1, \ldots, z_n) = 0$). Is it true that

$$D(z_1, \ldots, z_n)\alpha(z_1, \ldots, z_n) = o(1/n^{3/2})? \tag{15}$$

It is almost trivial that **(15)** holds if $c/n^{3/2}$ replaces $o(1/n^{3/2})$. Perhaps in fact $o(1/n^{3/2})$ can be replaced by c/n^2. This conjecture is perhaps too optimistic. The regular polygon shows that if true, it is best possible. **(15)** if true may throw some light on Heilbronn's problem.

V. T. Sós, E. Straus, and I slightly modified Heilbronn's problem as follows: Denote by $L(z_1, \ldots, z_n)$ the largest integer for which there are $L(z_1, \ldots, z_n)$ z's on a line. Assume $L(z_1, \ldots, z_n) = o(n^{1/2})$. Is it then true that there are three z's, $z_{i_1}, z_{i_2}, z_{i_3}$, not on a line for which the area of the triangle $(z_{i_1}, z_{i_2}, z_{i_3})$ is $o(1/n)$? Perhaps this conjecture is too optimistic. Pintz proved that if we assume $L(z_1, \ldots, z_n) < cn^\alpha$ for some fixed positive α then the conjecture indeed holds. The lattice points show that $o(n^{1/2})$ can certainly not be replaced by $O(n^{1/2})$.

V

Euclidean Ramsey Problems

A finite subset C of E_n is called r-Ramsey for E_n if for any partition of E_n into r sets S_i, $\bigcup_{i=1}^{r} S_i = E_n$, some S_i always contains a subset C' which is congruent to C. If C is r-Ramsey for every r for some E_n, then it is called Ramsey.

The study of these problems was started by "us" (Graham, Montgomery, Rothschild, Spencer, Straus, and myself) a few years ago. Many very interesting and challenging problems remain on this subject. We prove that if C is Ramsey, then C must lie on the surface of some sphere. Further, we prove that any subset of the vertices of a rectangular parallelepiped is Ramsey. We do not know which (if any) of these alternatives characterize Ramsey sets, and I offer \$500 for an answer to this question.

Is it true that every nonequilateral triangle is 2-Ramsey in the plane? I offer \$250 for a proof or disproof. L. Shader proved that all right triangles are 2-Ramsey (in E_2).

Gurevich asked: Put $E_n = \bigcup_{i=1}^{r} S_i$. Is it then true that at least one S_i contains the vertices of a k-dimensional simplex of volume 1? Graham proved and generalized this. His paper refers to nearly all the relevant literature on Euclidean Ramsey problems. (See Graham's paper in this volume.)

We posed the following problem: Let $S \subset E_2$ be such that no two points of S have distance 1. Is it then true that $\bar{S}$ (the complement of S) contains the vertices of a unit square? We could not settle this problem. Juhász proved that the answer is affirmative. In fact, she proved that our theorem remains true if "unit square" is replaced by "arbitrary four point configuration." She further proved that the theorem certainly fails if four is replaced by 12. Many further problems and conjectures are discussed in our papers.

References for Section V: [18, 19].

VI

In this final section, I discuss miscellaneous problems.

1. Let $x_1, \ldots, x_n$ be n points in the plane not all on a line. Denote by $A(x_1, \ldots, x_n)$ and $a(x_1, \ldots, x_n)$ the area of the largest and the smallest nonzero, respectively, of the triangles (x_i, x_j, x_l), $1 \leq i < j < l \leq n$. Purdy, Straus, and I proved that

$$A(x_1, \ldots, x_n)/a(x_1, \ldots, x_n) \geq [n/2] \qquad \textbf{(16)}$$

and we determined all cases when there is equality in **(16)**. Straus extended **(16)** to higher dimensions. In **(16)**, one could perhaps replace triangles by polygons with r vertices, but as far as I know, this has not yet been investigated.

Our paper on this subject will soon appear in *Discrete Mathematics*.

2. Corrádi, Hajnal, and I asked: Is it true that n points in the plane not all on a line determine at least $n - 2$ different angles? This is trivial if no three of the points are on a line, but seems to present curious difficulties in the general case and all we proved was that the number of distinct angles is $> cn^{1/2}$.

A related question is: Is it true that the smallest (nonzero) angle determined by our points is $\leq \pi/n$? We have equality for the regular polygon. This is again trivial if no three of the points are on a line, but in the general case we only get $c/n^{1/2}$.

An old problem of G. Dirac states: Is it true that if we join any two of our points, then there always is a point that is joined to at least $n/2 - c$ distinct lines? As far as I know, here also only $cn^{1/2}$ has been proved so far.

[**Editors' note:** A version of this problem has recently been solved by Szemerédi and Trotter, and independently by Beck.]

Scott asked: Is it true that our points determine at least $2[n/2]$ distinct directions? We have equality for the regular polygon. Scott proved $cn^{1/2}$. Here the problem was completely solved. First Burton and Purdy [20] proved $n/2$ and recently P. Ungár has proven $2[n/2]$, thus brilliantly settling the conjecture of Scott. It does not happen too often that a problem in this subject gets a complete solution.

[**Editors' note:** Ungár's paper has appeared in the *Journal of Combinatorial Theory, Series A* **33:** 343–347 (1982); see Jamison's paper in this volume for a summary of Ungár's proof as well as a history of the problem.]

The three problems discussed here are really unrelated.

3. A few years ago, I asked the following question: Let $f(n)$ be the largest integer for which there are n points $x_1, \ldots, x_n$ in E_2 so that there should be $f(n)$ distinct circles of unit radius passing through three of them. I observed

$$3n/2 < f(n) < n(n-1). \tag{17}$$

I believe that $f(n)/n \to \infty$ but $f(n)/n^2 \to 0$, but have made no progress on this problem.

It might be difficult to give an exact (or even an asymptotic) formula for $f(n)$. Several related questions can be posed.

Straus and I observed that if $n_k \geq k + 2\binom{k-1}{2}\binom{k-1}{3}$ and $x_1, \ldots, x_{n_k}$ are in general position (i.e., no three on a line and no four on a circle), then there are always k of them so that the $\binom{k}{3}$ radii of the circumscribed circles are all different. Probably our estimate for n_k is far from being best possible. See [21].

4. Denote by n_k the smallest integer (if it exists) for which every set of n_k points (no three on a line) contains k, which form a convex polygon that contains none of the other points in its interior. It is easy to see that $n_4 = 5$. Ehrenfeucht and Harborth proved that n_5 is finite. Harborth in fact proved $n_5 = 10$ [22]. It is not yet known if n_6 exists. J. D. Horton just informed me that he showed that n_7 does not exist. His proof will be published in the *Canadian Mathematical Bulletin*.

I raised this problem as a sharpening of an old problem of E. Klein (Mrs. Szekeres): Let $f(k)$ be the smallest interior for which every set of $f(k)$ points, no three on a line, contains the vertices of a convex k-gon. Klein observed 50 years ago that $f(4) = 5$ and Szekeres conjectured $f(k) = 2^{k-2} + 1$. Makai and Turán proved this for $k = 5$. Szekeres and I proved

$$2^{k-2} + 1 \leq f(k) \leq \binom{2k-4}{k-2}$$

This problem is discussed and referenced in [1].

5. Let $x_1, \ldots, x_n$ be n points in k-dimensional space. Assume that $d(x_i, x_j) \geq 1$ and that either two distances are equal or they differ by at least 1, i.e.,

$$d(x_i, x_j) = d(x_k, x_l) \qquad \text{or} \qquad |d(x_i, x_j) - d(x_k, x_l)| \geq 1. \tag{18}$$

Let

$$D_k(n) = \min_{x_1, \dots, x_n} D(x_1, \dots, x_n),$$

where $D(x_1, \dots, x_n)$ is the diameter of $x_1, \dots, x_n$ and the minimum is to be taken over all sets $x_1, \dots, x_n$ in E_k satisfying (18). Trivially $D_1(n) = n - 1$; I expect that $D_2(n) > cn$, but I am very far from being able to prove this. Kanold proved $D_2(n) > cn^{3/4}$ (*Elemente der Mathematik*, 1982, Manh) and the conjecture (4) would imply $D_2(n) > cn/(\log n)^{1/2}$; I find the conjecture $D_2(n) > cn$ a challenging and interesting problem and offer \$100 for a proof or disproof.

The lattice points in E_3 (respectively, E_k) easily show that

$$D_3(n) < c_3 n^{2/3}, \qquad D_k(n) < c_k n^{(k-1)/k}. \tag{19}$$

I would expect (without any real evidence) that the order of magnitude in (19) is best possible.

6. Let $x_1, \dots, x_n$ be n points in E_2, at most k on a line. It is easy to see that if $k\binom{r}{2} < n$, then there are always r of them no three of which are on a line; i.e., there are $(1 + o(1))(2n/k)^{1/2}$ points no three of them on a line. Probably this can be very much improved, but I have no further results on this problem. The best upper bound is the trivial $2n/k$.

7. It is not difficult to prove that if S is a set in the plane of infinite two-dimensional measure, then it contains for every c the vertices of a triangle of area c. I could never prove that there is an absolute constant α so that every planar set of measure $> \alpha$ contains the vertices of a triangle of area 1. In fact, perhaps $\alpha = \pi/\sqrt{3}$ (the equilateral triangle inscribed in the circle of area $\pi/\sqrt{3}$ has area 1). This conjecture is perhaps again a bit too optimistic.

Let R be large and S a set of plane measure $> cR^2$ in a circle of radius R. Is it true that S contains the vertices of an equilateral triangle of side > 1? Straus thought that this result perhaps already holds if we only assume that the area of S is greater than C, where C is a sufficiently large absolute constant.

An old conjecture of mine states: Let E be an infinite set of real numbers; then there always is a set S of positive linear measure that contains no subset similar (in the sense of elementary geometry) to E. I offer \$100 for a proof or disproof.

8. Two-distance sets and points in general position.

Denote by $g_k(n)$ the smallest integer for which every set of $g_k(n)$ points in E_n determines at least $k + 1$ distinct distances. Trivially $g_1(n) = n + 2$ and Blockhuis recently proved (sharpening a previous result of Delsarte, Goedels, and Seidel) that

$$g_2(n) \le \frac{(n+1)(n+2)}{2}.$$

Denote by $G_k(n)$ the smallest integer for which every set of $G_k(n)$ points in E_n contains a subset of $k + 1$ points any two distances of which are distinct; i.e., any set of $G_k(n)$ points contains a subset of $k + 1$ points that determines $\binom{k+1}{2}$ distinct distances. I proved long ago that $G_2(2) = 7$ and Croft proved $G_2(3) = 9$. Bárány and Füredi proved

$$G_2(n) < n^{\log n}.$$

Recently Blockhuis proved

$$G_2(n) < c_1 n^2.$$

It would be interesting to determine the exact values of $g_2(n)$ and $G_2(n)$ and more generally $g_k(n)$ and $G_k(n)$, for $k > 2$. This is not easy even for E_2.

The 2^n vertices of an n-dimensional cube determine $n + 1$ distinct distances. Is it true that every set of 2^n points in E_n determines at least cn distinct distances, where c is an absolute constant independent of n?

9. To end the paper, I state two problems from the collection of W. Moser. Problem 60 (due to Sierpinski) states: Do there exist two point sets in the plane such that no matter how they are placed in the plane, their intersection contains exactly one point? I proved the existence of two such sets by transfinite induction.

Problem 59 due to Steinhaus states: Does there exist a point set (in the plane) such that no matter how it is placed on the plane, it covers exactly one lattice point? I found this old problem of Steinhaus very challenging and got nowhere with it. More generally, let S be a set of positive numbers. Is there a set P in the plane such that no two points of P have distance in S and no matter how P is placed in the plane it covers exactly one lattice point? In the problem of Steinhaus, S is the set of numbers $(u^2 + v^2)^{1/2}$.

Notes and References

1. Erdős, P. 1974. On some problems of elementary and combinatorial geometry. Annali di Mat. Ser. 4 **103:** 99–108.
2. Erdős, P. 1980. Some combinatorial problems in geometry. *In* Lecture Notes in Mathematics No. 792 (proceedings of a conference held in Haifa, Israel, 1979): 46–53. Springer-Verlag. West Berlin, FRG.
3. Erdős, P. 1981. Some applications of graph theory and combinatorial methods to number theory and geometry. Algebraic methods in graph theory. Coll. Math. Soc. J. Bolyai **25:** 131–148.
4. Hadwiger, H., H. Debrunner & V. Klee. 1964. Combinatorial Geometry in the Plane. Holt, Rinehart & Winston. New York, N.Y.
5. Erdős, P. 1967. On some applications of graph theory to geometry. Canad. J. Math. **19:** 968–971.
6. Hansen, S. Contributions to the Sylvester–Gallai theory. Preprint.
7. Burr, S. A., B. Grünbaum & N. J. A. Sloane. 1979. The orchard problem. Geom. Dedicata **2:** 397–424. This paper contains an extensive bibliography and many interesting historical and other remarks. Many further references can be found in Reference 1.
8. Erdős, P. 1972. On a problem of Grünbaum. Canad. Math. Bull. **15:** 23–25.
9. Erdős, P. & G. Purdy. 1978. Some combinatorial problems in the plane. J. Combin. Theory Ser. A **25:** 205–210.
10. de Bruijn, N. G. & P. Erdős. 1951. A colour problem for infinite graphs and a problem in the theory of relations. Indag. Math. **13:** 371–373.
11. Erdős, P. & M. Simonovits. 1980. On the chromatic number of geometric graphs. Ars Combin. **9:** 229–246.
12. Frankl, P. 1977. On families of finite sets no two of which intersect in a singleton. Bull. Austral. Math. Soc. **17:** 125–134.
13. Frankl, P. & R. M. Wilson. 1981. Intersection theorems with geometric consequences. Combinatorica **1:** 357–368.
14. Larman, D. G. & C. A. Rogers. 1972. The realization of distances within sets in Euclidean space. Mathematica **19:** 1–24.
15. Woodall, D. R. 1973. Distances realised by sets covering the plane. J. Combin. Theory Ser. A **14:** 187–200.
16. Wormald, N. 1970. A 4-chromatic graph with a special plane drawing. J. Austral. Math. Soc. Ser. A **28:** 1–8.

17. Komlós, J., T. Pintz & E. Szemerédi. 1982. A lower bound for Heilbronn's problem. J. London Math. Soc. (2) **25:** 13–24.
18. Graham, R. L. 1980. On partition of E_n. J. Combin. Theory Ser. A **28:** 89–94. This paper contains very extensive references to the literature.
19. Juhász, R. 1979. Ramsey type theorems in the plane. J. Combin. Theory Ser. A **27:** 152–160.
20. Burton, G. R. & G. Purdy. 1979. The directions determined by n points in the plane. J. London Math. Soc. (2) **20:** 109–114.
21. Erdös, P. Some problems on elementary geometry, I and II. Austral. Math. Soc. Gaz. **2** (1975): 2–3 and **5** (1978): 52–54.
22. Harborth, H. 1979. Konvexe Fünfecke in ebenen Punktmengen. Elem. Math. **33:** 116–118.