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PROBLEMS AND RESULTS ON ADDITIVE PROPERTIES OF GENERAL SEQUENCES, IV.

by

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1. Let $A = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ be an infinite sequence of positive integers, put $A(n) = \sum_{\substack{a \in A \\ a \le n}} 1$, and for $n = 0, 1, 2, \dots$ let $R_1(n)$, $R_2(n)$, $R_3(n)$ denote the number of solutions of

(1) $\quad a_x + a_y = n$, $a_x \in A$, $a_y \in A$,

(2) $\quad a_x + a_y = n$, $x < y$, $a_x \in A$, $a_y \in A$

and

(3) $\quad a_x + a_y = n$, $x \leqq y$, $a_x \in A$, $a_y \in A$,

respectively.

In Parts I and II (see [4] and [5]) Erdős and Sárközy studied the regularity properties of the function $R_1(n)$. In Part III [5], the authors of this paper showed that under certain assumptions on A, $R_1(n+1) - R_1(n)$ cannot be bounded. <u>The aim of this paper is to study the monotonity properties of the functions $R_1(n)$, $R_2(n)$ and $R_3(n)$, respectively</u>. (See [2], [3] and [7] for other related results and problems.)

First we will determine those sequences A for which the function $R_1(n)$ is monotonous increasing from a certain point onwards.

THEOREM 1.

The function $R_1(n)$ is monotonous increasing from a certain point onwards, i.e., there exists an integer n_0 with

(4) $$R_1(n+1) \geq R_1(n) \quad \text{for} \quad n \geq n_0$$

if and only if the sequence A contains all the integers from a certain point onwards, i.e., there exists an integer n_1 with

(5) $$A \cap \{n_1, n_1+1, n_1+2, \ldots\} = \{n_1, n_1+1, n_1+2, \ldots\} .$$

It is, perhaps, somewhat surprising that the behaviour of the function $R_2(n)$ is different. Namely, there are much more sequences A for which $R_2(n)$ is monotonous increasing. In fact, we will show that

THEOREM 2.

Let $B = \{b_1, b_2, \ldots\}$ $(b_1 < b_2 < \ldots)$ be a sequence of positive integers such that

(i) B is a "Sidon-sequence", i.e.,

$b_x + b_y = b_u + b_v$, $b_x \in B$, $b_y \in B$, $b_u \in B$, $b_v \in B$, $b_x \leq b_y$, $b_u \leq b_v$ imply that $b_x = b_u$ and $b_y = b_v$,

(ii) all the elements of B are even,

and

(iii) $b_x \in B$, $b_y \in B$ imply that $(b_x + b_y)/2 \notin B$.

Then the complement of B, i.e., the sequence

(6) $$A = \{1, 2, \ldots, n, \ldots\} - B$$

is such that the function $R_2(n)$ is monotonous increasing:

(7) $$R_2(n+1) \geq R_2(n) \quad \text{for} \quad n = 1, 2, \ldots$$

(Probably, a similar construction could be given also for $R_3(n)$ in place of $R_2(n)$; unfortunately, we have not been able to give such a construction.)

By using the greedy algorithm, it can be shown easily that there exists an infinite sequence B such that

$$B(n) = \sum_{\substack{b \in B \\ b \leq n}} 1 \gg n^{1/3} \quad \text{(for all } n\text{)}$$

and it satisfies (i), (ii), (iii) and (iv) in Theorem 2. In this way, we obtain the existence of a sequence A such that

$$A(n) < n - cn^{1/3}$$

for large n and $R_2(n)$ is monotonous increasing from a certain point onwards.

On the other hand, we conjecture that if

$$A(n) = o(n)$$

then $R_2(n)$ and $R_3(n)$ cannot be monotonous increasing (from a certain point onwards). In fact, perhaps, it is enough to assume that

$$\lim_{n \to +\infty} \inf \frac{A(n)}{n} < 1$$

holds. Unfortunately, we have not been able to prove this. Instead, we will prove the following slightly weaker assertion:

THEOREM 3. If

(8) $$A(n) = o\left(\frac{n}{\log n}\right) ,$$

then the functions $R_2(n)$ and $R_3(n)$ cannot be monotonous increasing from a certain point onwards, i.e., for $j=2$ or 3, there does not exist an integer n_0 such that

(9) $$R_j(n+1) \geq R_j(n) \quad \text{for} \quad n \geq n_0 .$$

It is worth to note that by using the method of the proof of Theorem 3, we could study the more general problem when we count the solutions of (1), (2) and (3), respectively, with certain weights, i.e., we study the monotonity properties of the functions

$$\sum_{j=1}^{n-1} \alpha_j \alpha_{n-j} , \quad \sum_{j<n/2} \alpha_j \alpha_{n-j} \quad \text{and} \quad \sum_{j \leq n/2} \alpha_j \alpha_{n-j} ,$$

respectively, where $\alpha_1, \alpha_2, \dots$ are non-negative real numbers (satisfying certain assumptions).

2. PROOF OF THEOREM 1.

Assume that (5) holds and denote the complement of A *by* $B = \{b_1, b_2, \dots, b_t\}$:

$$B = \{b_1, b_2, \dots, b_t\} = \{1, 2, \dots\} - \{a_1, a_2, \dots\} .$$

(Clearly, (5) implies that B is finite.) Then for $n > 2b_t$ we we have

$$R_1(n) = \sum_{a_x + a_y = n} 1 = \sum_{i+j=n} 1 - 2 \sum_{b_x + a_y = n} 1 =$$

$$= (n-1) - 2 \sum_{b \in B} 1 = (n-1) - 2t$$

so that, obviously, $R_1(n)$ is monotonous increasing for $n > 2b_t$.

Assume now that for some n_0 , (4) holds. We have to show that this implies the existence of an integer n_1 satisfying (5). In order to prove this, we start out from the following trivial fact (which was used also by Dirac in [1]): if $n/2 \in A$

then $R_1(n)$ is even while if $n/2 \in A$ then $R_1(n)$ is odd. (In case of the functions $R_2(n)$ and $R_3(n)$, such an assertion does not hold. This is the reason of that that the study of the functions $R_2(n)$ and $R_3(n)$ is more difficult.) Thus (4) implies that for $2a_k > n_0$ we have

$$R_1(2a_k+1)-R_1(2a_k-1) = (R_1(2a_k+1)-R_1(2a_k))+(R_1(2a_k)-R_1(2a_k-1)) \geq$$
$$\geq 1+1 = 2 ,$$

hence for $n>n_0$,

$$R_1(n) \geq R_1(2[(n-1)/2]+1) \geq \sum_{n_0/2<j\leq[(n-1)/2]} (R_1(2j+1)-R_1(2j-1)) \geq$$

$$\geq \sum_{\substack{n_0/2<j\leq[(n-1)/2] \\ j\in A}} (R_1(2j+1)-R_1(2j-1)) \geq \sum_{\substack{n_0/2<j\leq[(n-1)/2] \\ j\in A}} 2 =$$

$$= 2(A([(n-1)/2])-A(n_0/2)) \geq 2(A(n/2)-1-A(n_0/2)) = 2A(n/2)-c_1$$

(where c_1 depends on A and n_0 but it is independent of n) so that

$$(10) \qquad R_1(n) > 2A(n/2) - c_2 \quad \text{for} \quad n = 1,2,\dots .$$

Let $B = \{b_1,b_2,\dots\}$ $(b_1<b_2<\dots)$ denote the complement of A:

$$B = \{b_1,b_2,\dots\} = \{1,2,\dots\}- A ,$$

and put

$$B(n) = \sum_{\substack{b\leq n \\ b\in B}} 1 .$$

CASE 1.

Assume first that

$$\limsup_{n\to+\infty} (B(2n)-B(n)) = +\infty .$$

Let us write

$$\rho = \liminf_{n \to +\infty} \frac{A(n)}{n} .$$

Let us define an infinite sequence $N_1 < N_2 < \dots$ of positive integers in the following way:

(11) if $\rho = 0$ then let $\lim_{k \to +\infty} \frac{A(2N_k)}{2N_k} = 0$,

(12) if $\rho > 0$ then let $\lim_{k \to +\infty} (B(2N_k) - B(N_k)) = +\infty$.

Let k be a large integer, put

$$\max_{N_k < n \leqq 3N_k} \sum_{\substack{a+b=n \\ a \leqq N_k < b \leqq 2N_k \\ a \in A,\ b \in B}} 1 = M$$

and let N be an integer with $N_k < N \leqq 3N_k$ for which this maximum is assumed:

$$\sum_{\substack{a+b=N \\ a \leqq N_k < b \leqq 2N_k \\ a \in A,\ b \in B}} 1 = M .$$

Then we have

$$2N_k M = \sum_{N_k < n \leqq 3N_k} M \geq$$

$$\geq \sum_{N_k < n \leqq 3N_k} \sum_{\substack{a+b=n \\ a \leqq N_k < b \leqq 2N_k \\ a \in A,\ b \in B}} 1 = \sum_{\substack{a \leqq N_k < b \leqq 2N_k \\ a \in A,\ b \in B}} 1 =$$

$$= \left(\sum_{\substack{a \leqq N_k \\ a \in A}} 1 \right) \left(\sum_{\substack{N_k < b \leqq 2N_k \\ b \in B}} 1 \right) = A(N_k)(B(2N_k) - B(N_k))$$

hence

$$(13) \qquad M \geq \frac{1}{2}\,\frac{A(N_k)(B(2N_k)-B(N_k))}{N_k}\ .$$

On the other hand, in view of (10) we have

$$2A(N/2)-c_2<R_1(N) = \sum_{\substack{a_x+a_y=N\\ a_x\in A,\, a_y\in A}} 1 \leq 2 \sum_{\substack{a_x+a_y=N\\ a_x\in A,\, a_y\in A\\ a_x\leq N/2}} 1 =$$

$$= 2\left(\sum_{\substack{a_x\in A\\ a_x\leq N/2}} 1 - \sum_{\substack{a_x\in A,\ N-a_x\notin A\\ a_x\leq N/2}} 1\right) =$$

$$= 2\left(A(N/2) - \sum_{\substack{a+b=N\\ a\in A,\, b\in B\\ a<b}} 1\right) \leq$$

$$\leq 2\left(A(N/2) - \sum_{\substack{a+b=N\\ a\in A,\, b\in B\\ a\leq N_k<b\leq 2N_k}} 1\right) = 2(A(N/2)-M)$$

hence

$$(14) \qquad M < \frac{1}{2}\,c_2\ .$$

(13) and (14) yield that

$$(15) \qquad c_2 > \frac{A(N_k)(B(2N_k)-B(N_k))}{N_k}\ .$$

By (11) and (12), for $\rho=0$ and $k\to+\infty$ we have

$$\frac{A(N_k)(B(2N_k)-B(N_k))}{N_k} = \frac{A(N_k)(N_k-A(2N_k)+A(N_k))}{N_k} \geq$$

$$\geq \frac{A(N_k)(N_k-A(2N_k))}{N_k} > \frac{A(N_k)\cdot N_k/2}{N_k} = \frac{A(N_k)}{2} \to +\infty$$

while for $\rho>0$ and $k\to+\infty$,

$$\frac{A(N_k)(B(2N_k)-B(N_k))}{N_k} > \frac{c}{2}(B(2N_k)-B(N_k)) \to +\infty .$$

In either case, (15) cannot hold for large k, and this contradiction shows that in Case 1, $R_1(n)$ cannot be monotonous increasing.

CASE 2.

Assume now that

(16) $$\limsup_{n\to+\infty} (B(2n)-B(n)) < +\infty$$

and assume first that $\mathcal{B}$ is infinite:

(16) implies that there exist constants $L>0$, $\delta>0$ and j_0 such that

(17) $$B(2x) - B(x) < L \quad \text{for} \quad x>0$$

and

$$B(n) = O(\log n)$$

so that

(18) $$b_j > (1+\delta)^j \quad \text{for} \quad j>j_0 :$$

Furthermore, (16) implies that if t is large enough then

(19) $$b_{t(j+1)} > 2b_{tj} .$$

Let us fix such an integer j, and let k be a large integer. For $j=1,2,\dots,k^2$, put

$$N_j = b_{tk^3} + b_{t(k^2+j)} .$$

We are going to show that there exists an integer j such that

(20) $$1 \le j \le k^2-1 \quad \text{and} \quad \mathcal{B} \cap (N_j/2,\ N_{j+1}/2] = \emptyset .$$

In fact, if such an integer j does not exist then we have

$$(21) \quad B(N_{k^2}/2)-B(N_1/2) = \sum_{j=1}^{k^2-1} (B(N_{j+1}/2)-B(N_j/2)) \geq \sum_{j=1}^{k^2-1} 1 = k^2-1$$

Here

$$\frac{N_1}{2} = \frac{b_{tk^3}+b_{t(k^2+j)}}{2} > \frac{1}{2} b_{tk^3} = \frac{1}{4} (b_{tk^3}+b_{tk^3}) >$$

$$> \frac{1}{4}(b_{tk^3}+b_{2tk^2}) = \frac{1}{4} N_{k^2}$$

so that

$$B(N_1/2) \geqq B(N_{k^2}/4) ,$$

$$(22) \qquad B(N_{k^2}/2)-B(N_{k^2}/4) \geq B(N_{k^2}/2)-B(N_1/2) .$$

(21) and (22) yield that

$$B(N_{k^2}/2)-B(N_{k^2}/4) \geq k^2 - 1 .$$

But by (17), this inequality cannot hold for large k which proves the existence of an integer j satisfying (20).

Let us fix such an integer j. We are going to show by induction on i that for $i = N_j, N_{j+1}, \dots, N_{j+1}-1$,

(23) i can be written in the form b_x+b_y with $b_x \in B$, $b_y \in B$, $x \neq y$

For $i=N_j$, we have

$$i=N_j = b_{tk^3} + b_{t(k^2+j)}$$

so that (23) holds.

Assume now that (23) holds for some $N_j \leq i < N_{j+1}-1$. Then we have

(24) $R_1(i+1)-R_1(i) =$

$$= \left(\sum_{u+v=i+1} 1 - \sum_{\substack{u+v=i+1 \\ u\in B}} 1 - \sum_{\substack{u+v=i+1 \\ v\in B}} 1 + \sum_{\substack{u+v=i+1 \\ u\in B, v\in B}} 1 \right) -$$

$$- \left(\sum_{u+v=i} 1 - \sum_{\substack{u+v=i \\ u\in B}} 1 - \sum_{\substack{u+v=i \\ v\in B}} 1 + \sum_{\substack{u+v=i \\ u\in B, v\in B}} 1 \right) =$$

$$= \left((i+1) - 2 \sum_{\substack{u+v=i+1 \\ u\in B}} 1 + \sum_{\substack{u+v=i+1 \\ u\in B,\ v\in B}} 1 \right) -$$

$$- \left(i-2 \sum_{\substack{u+v=i \\ u\in B}} 1 + \sum_{\substack{u+v=i \\ u\in B, v\in B}} 1 \right) =$$

$$= 1-2(B(i)-B(i-1)) + \sum_{\substack{u+v=i+1 \\ u\in B, v\in B}} 1 - \sum_{\substack{u+v=i \\ u\in B, v\in B}} 1 \leq$$

$$\leq 1+ \sum_{\substack{u+v=i+1 \\ u\in B, v\in B}} 1 - 2 = -1 + \sum_{\substack{u+v=i+1 \\ u\in B, v\in B}} 1 .$$

By (4), we have

(25) $$R_1(i+1) - R_1(i) \geq 0$$

(provided that k is large so that $i \geq N_j > b_{tk^2} > n_0$). (24) and (25) imply that

$$\sum_{\substack{u+v=i+1 \\ u\in B, v\in B}} 1 \geq 1$$

so that $i+1$ can be written in the form

$$b_x + b_y = i+1 \ , \ b_x \in B \ , \ b_y \in B \ .$$

Furthermore, by (20), here we cannot have $b_x=b_y$ which proves that (23) holds also with $i+1$ in place of i.

By (18), (19) and (23) we have

$$(26)\qquad \sum_{i=N_j}^{N_{j+1}-1} \sum_{\substack{b_x+b_y=i \\ b_x\in B, b_y\in B}} 1 \geq \sum_{i=N_j}^{N_{j+1}-1} 2 = 2(N_{j+1}-N_j) =$$

$$= 2(b_{t(k^2+j+1)} - b_{t(k^2+j)}) > 2(b_{t(k^2+j+1)} - \tfrac{1}{2} b_{t(k^2+j+1)}) =$$

$$= b_{t(k^2+j+1)} > (1+\delta)^{t(k^2+j+1)} .$$

On the other hand, by (17) we have

$$(27)\qquad \sum_{i=N_j}^{N_{j+1}-1} \sum_{\substack{b_x+b_y=i \\ b_x\in B, b_y\in B}} 1 = \sum_{\substack{N_j\leq b_x+b_y<N_{j+1} \\ b_x\in B,\ b_y\in B}} 1 \leq$$

$$\leq \sum_{\substack{b_x+b_y\leq N_{j+1} \\ b_x\in B, b_y\in B}} 1 \leq \sum_{\substack{b_x\leq N_{j+1}, b_y\leq N_{j+1} \\ b_x\in B,\ b_y\in B}} 1 = \left\{\sum_{\substack{b_x\leq N_{j+1} \\ b_x\in B}} 1\right\}^2 =$$

$$= (B(N_{j+1}))^2 = (B(b_{tk^3}+b_{t(k^2+j)}))^2 \leq (B(2b_{tk^3}))^2 =$$

$$= (B(b_{tk^3})+(B(2b_{tk^3})-B(b_{tk^3})))^2 < (tk^3+L)^2$$

(26) and (27) imply that

$$(1+\delta)^{t(k^2+j+1)} < (tk^3+L)^2 .$$

But for large k, this inequality cannot hold. This contradiction proves that if $R_1(n)$ is monotonous increasing, then the sequence B cannot be infinite.

Thus (4) implies that B must be finite which is equivalent to (5) and this completes the proof of Theorem 1.

3. PROOF OF THEOREM 2.

Let

$$B(n) = \sum_{\substack{b\in B\\ b\leq n}} 1 ,$$

$$\eta(i) = \begin{cases} 1 & \text{if } i\in B \\ 0 & \text{if } i\notin B \end{cases}$$

and let us denote the number of solutions of

$$b_x + b_y = n , \; b_x\in B , \; b_y\in B , \; x<y$$

by $R^*(n)$.

Then we have

$$(28) \quad R_2(n) = \sum_{\substack{a_x+a_y=n\\ a_x\in A,\ a_y\in A\\ x<y}} 1 = \sum_{1\leq i<n/2} (1-\eta(i))(1-\eta(n-i)) =$$

$$= \sum_{1\leq i<n/2} 1 - \sum_{\substack{1\leq i<n/2\\ i\in B,\ i\neq n/2}} 1 + R^*(n) =$$

$$= \begin{cases} k-1-B(k-1)+R^*(2k) & \text{for } n=2k , \\ k-B(k)+R^*(2k+1) & \text{for } n=2k+1 . \end{cases}$$

Thus for $k=1,2,\dots$ we have

$$(29) \quad R_2(2k+1)-R_2(2k)=1-(B(k)-B(k-1))+R^*(2k+1)-R^*(2k) \geq$$

$$\geq 1-(B(k)-B(k-1))-R^*(2k) .$$

Clearly,

$$B(k) - B(k-1) \leq 1 ,$$

and by (i),

$$R^*(2k) \leq 1 .$$

Furthermore, by (iii),

$$B(k)-B(k-1) = 1$$

and

$$R^*(2k) = 1$$

cannot hold simultaneously so that

$$(B(k)-B(k-1)) + R^*(2k) \leq 1 .$$

Thus we obtain from (29) that

$$R_2(2k+1)-R_2(2k) \geq 1-1 = 0 \quad (\text{for } k=1,2,\dots) . \tag{30}$$

Similarly, we get from (28) that

$$R_2(2k)-R_2(2k-1) = R^*(2k)-R^*(2k-1) \geq R^*(2k) \geq 0 \tag{31}$$

since $R^*(2k-1) = 0$ by (ii).

(30) and (31) yield (7) which completes the proof of Theorem 2.

4. PROOF OF THEOREM 3.

We start out from the indirect assumption that (8) holds, however, for $j=2$ or 3 and for some integer n_0, (9) holds.

First we show that there exist infinitely many integers N satisfying

$$A(N+j) < A(N)\left(\frac{N+j}{N}\right)^2 \quad \text{for } j=1,2,\dots . \tag{32}$$

In fact, if (32) holds for finitely many N, then there exists an integer N_0 such that

$$A(N_0) > 1$$

and for $N \geq N_0$, there exists an integer $N' = N'(N)$ satisfying $N' > N$ and

$$A(N') \geq A(N)\left(\frac{N'}{N}\right)^2 .$$

Then we get (by induction) that there exist integers $N_1 < N_2 < \ldots < N_j < \ldots$ such that

$$A(N_{j+1}) \geq A(N_j)\left(\frac{N_{j+1}}{N_j}\right)^2 \quad \text{for} \quad j=0,1,2,\ldots ,$$

hence

$$(33) \qquad A(N_{k+1}) = A(N_0)\prod_{j=0}^{k}\frac{A(N_{j+1})}{A(N_j)} \geq A(N_0)\prod_{j=0}^{k}\left(\frac{N_{j+1}}{N_j}\right)^2 =$$

$$= A(N_0)\left(\frac{N_{k+1}}{N_0}\right)^2 > \left(\frac{N_{k+1}}{N_0}\right)^2 > N_{k+1}^{3/2}$$

for large enough k. On the other hand, clearly we have

$$(34) \qquad A(N_{k+1}) = \sum_{\substack{a \in A \\ a \leq N_{k+1}}} 1 \leq \sum_{a \leq N_{k+1}} 1 = N_{k+1}$$

(33) and (34) cannot hold simultaneously and this contradiction proves the existence of infinitely many integers N satisfying (32).

Now we are going to estimate $R_j(n)$ in terms of $A(2n)$. In view of (9), for $n \geq n_0$ we have

$$(A(2n))^2 = \left(\sum_{\substack{a_i \in A \\ a_i \leq 2n}} 1\right)^2 = \sum_{\substack{a_i \in A, a_j \in A \\ a_i \leq 2n, a_j \leq 2n}} 1 \geq$$

$$\geq \sum_{\substack{a_i+a_j\leq 2n \\ a_i\in A, a_j\in A'}} 1 \geq \sum_{i=1}^{2n} R_j(i) \geq \sum_{i=n+1}^{2n} R_j(i) \geq \sum_{i=n+1}^{2n} R_j(n) = nR_j(n)$$

hence

$$R_j(n) \leq \frac{(A(2n))^2}{n} \quad \text{for} \quad n \geq n_0 . \tag{35}$$

Furthermore, by (9), for large n we have $R_j(n) \geq 1$ so that we obtain from (35) that

$$(A(2n))^2 \geq n \quad \text{for large} \quad n . \tag{36}$$

5. Throughout the remaining part of the proof of Theorem 3, we use the following notations:

N denotes a large integer satisfying (32). We write $e^{2-i\alpha} = e(\alpha)$, and we put $r = e^{-1/N}$, $z = re(\alpha)$ where α is a real variable (so that a function of form $p(z)$ is a function of the real variable α : $p(z) = p(re(\alpha)) = P(\alpha)$). We write

$$f(z) = \sum_{j=1}^{+\infty} z^{a_j} .$$

(By $r<1$, this infinite series and all the other infinite series in the remaining part of the proof are absolutely convergent.) Then we have

$$\tfrac{1}{2}((f(z))^2-f(z^2)) = \sum_{n=1}^{+\infty} R_2(n)z^n$$

and

$$\tfrac{1}{2}((f(z))^2+f(z^2)) = \sum_{n=1}^{+\infty} R_3(n)z^n$$

so that for both $j=2$ and $j=3$,

$$|f(z)|^2-|f(z^2)| \leq 2\left|\sum_{n=1}^{+\infty} R_j(n)z^n\right|$$

hence

$$(37)\quad \int_0^1 |f(z)|^2 d\alpha - \int_0^1 |f(z^2)| d\alpha \leqq 2 \int_0^1 \left|\sum_{n=1}^{+\infty} R_j(n)z^n\right| d\alpha .$$

Writing

$$J_1=\int_0^1 |f(z)|^2 d\alpha,\ J_2=\int_0^1 |f(z^2)| d\alpha \quad \text{and} \quad J=\int_0^1 \left|\sum_{n=1}^{+\infty} R_j(n)z^n\right| d\alpha ,$$

(37) can be rewritten in the form

$$(38)\qquad J_1 - J_2 \leqq 2J .$$

We will give a lower bound for J_1-J_2 and an upper bound for $2J$. The lower bound for J_1-J_2 will be greater than the upper bound for $2J \geqq J_1-J_2$. This contradiction will prove that the indirect assumption (9) cannot hold, and this will complete the proof of Theorem 3.

6. In this section, we give a lower bound for J_1-J_2. First we estimate J_1. By the Parseval formula, we have

$$(39)\qquad J_1 = \int_0^1 |f(z)|^2 d\alpha = \sum_{a\in A} r^{2a} \geqq \sum_{\substack{a\in A\\ a\leq N}} r^{2N} =$$

$$= e^{-2} \sum_{\substack{a\in A\\ a\leqq N}} 1 = e^{-2}A(N) > \frac{1}{10}A(N) .$$

Now we are going to estimate J_2. By (32), the Cauchy inequality and the Parseval formula, for large N we have

$$(40)\quad J_2 = \int_0^1 |f(z^2)|\,d\alpha = \left(\int_0^1 |f(z^2)|^2 d\alpha\right)^{1/2} = \left(\sum_{a\in A} r^{4a}\right)^{1/2} =$$

$$= \left\{(1-r^4)\left(\frac{1}{1-r^4}\sum_{a\in A} r^{4a}\right)\right\}^{1/2} = \left\{(1-r^4)\left(\sum_{n=1}^{+\infty} A(n) r^{4n}\right)\right\}^{1/2} =$$

$$= \left\{(1-r^4)\left(\sum_{n=1}^{N} A(n) r^{4n} + \sum_{n=N+1}^{+\infty} A(n) r^{4n}\right)\right\}^{1/2} <$$

$$< \left\{(1-r)(1+r+r^2+r^3) N A(N) + (1-r^4)\sum_{n=N+1}^{+\infty} A(N)\left(\frac{n}{N}\right)^2 r^{4n}\right\}^{1/2} <$$

$$< \left\{(1-e^{-1/N}) N A(N) + \frac{A(N)}{N^2}(1-r^4)\sum_{n=1}^{+\infty} n^2 r^{4n}\right\}^{1/2} <$$

$$< \left\{N^{-1}\cdot N A(N) + A(N) N^{-2}\sum_{n=1}^{+\infty}(n^2-(n-1)^2) r^{4n}\right\}^{1/2} <$$

$$< \left\{A(N) + A(N) N^{-2}\sum_{n=1}^{+\infty} 2n\cdot r^{4n}\right\}^{1/2} < \left\{A(N) + A(N) N^{-2}\cdot 2r^4 \sum_{n=1}^{+\infty} n r^{4(n-1)}\right\}^{1/2}$$

$$= \left\{A(N) + A(N) N^{-2}\cdot 2r^4 (1-r^4)^{-2}\right\}^{1/2} < \left\{A(N) + 2A(N) N^{-2}(1-r)^{-2}\right\}^{1/2} =$$

$$= \left\{A(N) + 2A(N) N^{-2}(1-e^{-1/N})^{-2}\right\}^{1/2} < \left\{A(N) + 2A(N) N^{-2}(1/2N)^{-2}\right\}^{1/2} =$$

$$= (9A(N))^{1/2} = 3(A(N))^{1/2}.$$

(39) and (40) yield for large N that

$$(41)\qquad J_1 - J_2 > \frac{1}{20}A(N) - 3(A(N))^{1/2} > \frac{1}{21}A(N).$$

7. In this section, we give an upper bound for J and we complete the proof of Theorem 3. We rewrite J in the following way:

$$(42)\quad J = \int_0^1 \Big|\sum_{n=1}^{+\infty} R_j(n)z^n\Big| d\alpha = \int_0^1 \Big|(1-z)\sum_{n=1}^{+\infty} R_j(n)z^n\Big| |1-z|^{-1}\, d\alpha\ .$$

In view of (9), (32) and (35), we have

$$(43)\qquad \Big|(1-z)\sum_{n=1}^{+\infty} R_j(n)z^n\Big| = \Big|\sum_{n=1}^{+\infty}(R_j(n)-R_j(n-1))z^n\Big| \leq$$

$$\leq \sum_{n=1}^{n_0}|R_j(n)-R_j(n-1)|r^n + \sum_{n=n_0+1}^{+\infty}|R_j(n)-R_j(n-1)|r^n <$$

$$< \sum_{n=1}^{n_0}|R_j(n)-R_j(n-1)| + \sum_{n=n_0+1}^{+\infty}|R_j(n)-R_j(n-1)|r^n =$$

$$= \sum_{n=1}^{n_0}|R_j(n)-R_j(n-1)| + \sum_{n=n_0+1}^{+\infty}(R_j(n)-R_j(n-1))r^n <$$

$$< 2\sum_{n=1}^{n_0}|R_j(n)-R_j(n-1)| + \sum_{n=1}^{+\infty}(R_j(n)-R_j(n-1))r^n =$$

$$= c_1 + \sum_{n=1}^{+\infty} R_j(n)(r^n-r^{n+1}) = c_1+(1-r)\sum_{n=1}^{+\infty}R_j(n)r^n <$$

$$< c_1+\sum_{n=1}^{n_0-1} R_j(n)+(1-r)\sum_{n=n_0}^{+\infty} R_j(n)r^n <$$

$$< c_2+(1-e^{-1/N})\left\{\sum_{n=n_0}^{N} R_j(N)+\sum_{n=N+1}^{+\infty}R_j(n)r^n\right\} <$$

$$< c_2+N^{-1}\left\{N\frac{(A(2N))^2}{N} + \sum_{n=N+1}^{+\infty}\frac{(A(2n))^2}{n}\, r^n\right\} <$$

$$< c_2+N^{-1}\left\{(A(N))^2\cdot\left(\frac{2N}{N}\right)^4 + \sum_{n=N+1}^{+\infty}\left\{A(N)\left(\frac{2n}{N}\right)^2\right\}^2\cdot\frac{1}{n}\, r^n\right\} <$$

$$< c_2+(A(N))^2\left\{16N^{-1}+4N^5\sum_{n=1}^{+\infty}n^3r^n\right\} =$$

$$= c_2+(A(N))^2\left\{16N^{-1}+4N^{-5}(1-r)^{-1}\sum_{n=1}^{+\infty}n^3(r^n-r^{n+1})\right\} =$$

$$= c_2+(A(N))^2\left\{16N^{-1}+4N^{-5}(1-r)^{-1}\sum_{n=1}^{+\infty}(n^3-(n-1)^3)r^n\right\} <$$

$$< c_2 + (A(N))^2\left(16N^{-1}+4N^{-5}(1-r)^{-1}\sum_{n=1}^{+\infty}4n^2r^n\right) =$$

$$= c_2+(A(N))^2\left(16N^{-1}+16N^{-5}(1-r)^{-2}\sum_{n=1}^{+\infty}n^2(r^n-r^{n+1})\right) =$$

$$= c_2+(A(N))^2\left(16N^{-1}+16N^{-5}(1-r)^{-2}\sum_{n=1}^{+\infty}(n^2-(n-1)^2)r^n\right) <$$

$$< c_2+(A(N))^2\left(16N^{-1}+32N^{-5}(1-r)^{-2}\sum_{n=1}^{+\infty}nr^{n-1}\right) =$$

$$= c_2+(A(N))^2(16N^{-1}+32N^{-5}(1-r)^{-4}) <$$

$$< c_2+(A(N))^2(16N^{-1}+32N^{-5}(1-e^{-1/N})^{-4}) <$$

$$< c_2+(A(N))^2(16N^{-1}+32N^{-5}(2N)^4) <$$

$$< c_2+600(A(N))^2N^{-1} < c_3(A(N))^2N^{-1} .$$

Furthermore, we have

(44) $$|1-z| =((1-z)(1-\bar{z}))^{1/2} = (1+|z|^2-2\text{Re } z)^{1/2} =$$

$$= (1+r^2-2r\cos 2\pi\alpha)^{1/2} = ((1-r)^2+2r(1-\cos 2\pi\alpha))^{1/2} >$$

$$> (2r(1-\cos 2\pi\alpha))^{1/2} = (2e^{-1/N}.2\sin^2\pi\alpha)^{1/2} \geq$$

$$\geq \left(2\cdot\frac{1}{2}\cdot 2(2\alpha)^2\right)^{1/2} = (8\alpha^2)^{1/2} \geq 2\alpha \quad \text{for} \quad 0 \leq \alpha \leq 1/2$$

and (for large N)

(45) $$|1-z| = ((1-r)^2+2r(1-\cos 2\pi\alpha))^{1/2} > ((1-r)^2)^{1/2} = 1-r = 1-e^{-1/N} >$$

$$> 1/2N \quad \text{for all } \alpha .$$

(42), (43), (44) and (45) yield that

$$(46) \qquad J \leq \int_0^1 c_3(A(N))^2 N^{-1} \cdot |1-z|^{-1}\, d\alpha =$$

$$= 2c_3(A(N))^2 N^{-1} \int_0^{1/2} |1-z|^{-1}\, d\alpha =$$

$$= c_4(A(N))^2 N^{-1}\left\{\int_0^{1/N} |1-z|^{-1}\, d\alpha + \int_{1/N}^{1/2} |1-z|^{-1}\, d\alpha\right\} <$$

$$< c_4(A(N))^2 N^{-1}\left\{\int_0^{1/N} 2N\, d\alpha + \int_{1/N}^{1/2} (2\alpha)^{-1} d\alpha\right\} <$$

$$< c_4(A(N))^2 N^{-1}(2+\log N) < c_5(A(N))^2 N^{-1} \log N .$$

By (38), (41) and (46), we have

$$\frac{1}{21} A(N) < J_1 - J_2 \leq 2J < c_6(A(N))^2 N^{-1} \log N$$

hence

$$c_7 \frac{N}{\log N} < A(N) .$$

By (8), this inequality cannot hold, so that the indirect assumption (9) leads to a contradiction which completes the proof of Theorem 3.

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