OFFPRINTS FROM GRAPH THEORY AND ITS APPLICATIONS TO ALGORITHMS AND COMPUTER SCIENCE

Edited by Y. Alavi et al.

PROBLEMS AND RESULTS ON CHROMATIC NUMBERS IN FINITE AND INFINITE GRAPHS

P. Erdös

Hungarian Academy of Science

During my long life I wrote many papers of similar title. To avoid repetitions and to shorten the paper I will discuss almost entirely recent problems and will not give proofs.

First of all I discuss some problems which came up during a recent visit to Calgary. An old problem in graph theory states that if G_1 and G_2 both have chromatic number $\geq k$ then $G_1 \times G_2$ also has chromatic number $\geq k$. The product $G_1 \times G_2$ is defined as follows: If $x_1, \cdots, x_u$; $y_1, \cdots, y_v$ are the vertices of G_1 and G_2, then (x_i, y_j), $1 \leq i \leq u$; $1 \leq j \leq v$ are the vertices of $G_1 \times G_2$. Join (x_i, y_j) to (x_{i_1}, y_{j_1}) if and only if x_i is joined to x_{i_1} and y_j to y_{j_1}. (Observe (x_i, y_j) and (x_{i_1}, y_{j_1}) are joined only if $i \neq i_1$ and $j \neq j_1$.)

This conjecture was known (and easy) for $k \leq 3$ and Sauer and El-Zahar proved it for $k = 4$ not long ago. The proof was surprisingly difficult and does not seem to generalize for $k > 4$. Hajnal proved that if G_1 and G_2 both have infinite chromatic number then their product also has infinite chromatic number.

Perhaps more surprisingly he showed that there are two graphs of chromatic number $\aleph_{k+1}$ whose product has chromatic number $\aleph_k$. The following two problems remain open: Are there two graphs of chromatic number $\aleph_{k+2}$ whose product has chromatic number $\le \aleph_k$ are there two graphs of chromatic number $\aleph_\omega$ whose product has chromatic number $< \aleph_\omega$? These problems are analogous to some old problems of Hajnal and myself. We proved [4] that for every α there is a graph of power $(2^{\aleph_\alpha})^+$ and chromatic number $\ge \aleph_{\alpha+1}$ so that every subgraph of power $\aleph \le 2^{\aleph_\alpha}$ has chromatic number $\aleph_\alpha$. We did not know (even assuming G.C.H.) if there is a graph of power and chromatic number $\aleph_{\alpha+2}$ so that each subgraph of power $\aleph_{\alpha+1}$ has chromatic number $\aleph_\alpha$. Recently Baumgartner proved that the existence of such a graph is consistent. In fact he proved that it is consistent with the generalised continuum hypothesis there is a graph of power and chromatic number $_2$ all of whose subgraphs of power $\le \aleph_1$ have chromatic number $\le \aleph_0$. At the moment it seems hopeless to find a graph of power and chromatic number $\aleph_3$ all of whose subgraphs of power $\le \aleph_3$ have chromatic number $\le \aleph_0$. Laver and Foreman showed that it is consistent (relative to the existence of a very large cardinal) that if every subgraph of power $\aleph_1$ of a graph of size has $\aleph_2$ chromatic number $\aleph_1$ then the whole graph has chromatic number $\le \aleph_1$. Thus it is consistent that our example is best possible.

Shelah showed that in the constructible universe for every regular K that is not weakly compact, there is a graph of size K and chromatic number $\aleph_1$ all of whose subgraphs of size $< K$ have chromatic number $\le \aleph_0$.

As far as we know, our old problem is still open: If G has power $\aleph_{\omega+1}$ and chromatic number $\aleph_1$, then it is consistent that it must hve a subgraph of power $\aleph_\omega$ and chromatic number $\aleph_1$.

An old theorem of Hajnal, Shelah and myself [5] states that if G has chromatic number $\aleph_1$, then there is an $n_0 = n_0(G)$

so that G contains a circuit C_n for every $n > n_0$. On the other hand, we know almost nothing of the 4-chromatic subgraphs that must be contained in G. In particular we do not know if G_1 and G_2 have chromatic number $\aleph_1$ whether there is an H of chromatic number 4 which is a subgraph of both G_1 and G_2. It seems certain that this is true and perhaps it remains true if 4 is replaced by any finite n and perhaps by $\aleph_0$. Hajnal, on the other hand, constructed $\aleph_1$ graphs G_α, $1 \le \alpha \le \omega_1$ of power $2^{\aleph_0}$ and chromatic number $\aleph_1$ no two of which contain a common subgraph H of chromatic number $\aleph_1$.

Now we have to state the fundamental conjecture of W. Taylor which, unfortunately, Hajnal and I missed (probably due to old age, stupidity and laziness): Let G be a graph of chromatic number $\aleph_1$. Is it then true that for every cardinal number m there is a graph G_m of chromatic number m all finite subgraphs of which are also subgraphs of G? No real progress has been made with this beautiful conjecture. Hajnal, Shelah and I investigated the following related problem: We call a family F of finite graphs good if there is an at least $\aleph_1$-chromatic graph G all whose finite subgraphs are in F. (We write at least $\aleph_1$-chromatic instead of $\aleph_1$-chromatic since Galvin [8] observed more than 15 years ago that it is not at all obvious that every graph of chromatic number greater than $\aleph_1$ contains a subgraph of chromatic number $\aleph_1$. In fact he proved that it is consistent that there is a graph of chromatic number $\aleph_2$ that does not contain an induced subgraph of chromatic number $\aleph_1$.) We call F very good if for every cardinal number m there is a graph G_m of chromatic number $\ge$ m all of whose finite subgraphs are in F. Hopefully good = very good. We observed that the set of all finite subgraphs of our [3] old r-shift graphs are very good for every r. The r-shift graph is defined as follows: Let $\{x_\alpha\}$ be a well ordered set. The vertices of the r-shift graph are the r-tuples

$\{x_{\alpha_1}, \cdots, x_{\alpha_r}\}$ $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Two such r-tuples $\{x_{\alpha_1}, x_{\alpha_2}, \cdots, x_{\alpha_r}\}, \{y_{\beta_1}, y_{\beta_2}, \cdots, y_{\beta_r}\}$ are joined if and only if $y_{\beta_1} = x_{\alpha_2}, \cdots, y_{\beta_{r-1}} = x_{\alpha_r}$.

We also stated the following problem: A family F_r of finite graphs is called r-good if there is a graph G_r of power $\leq \aleph_{r+1}$ and chromatic number $\geq \aleph_1$ all of whose finite subgraphs are in F_r. It is called r-very good if (for every cardinal $\aleph_\alpha$) there is a graph G of chromatic number $\leq \aleph_\alpha$ and power $\leq \aleph_{\alpha+r}$ all of whose finite subgraphs are in F_r. Hopefully r-good = r-very good. We proved that for $r < \omega$ $F_{r+1} \subset F_r$ and the inclusion is proper. We do not know what happens for $r > \omega$.

We proved that the number of vertices of an at least $\aleph_1$-chromatic graph all whose finite subgraphs are subgraphs of the r-th shift-graph must have power $\exp_r(\aleph_1)^+ = \aleph_{r+1}$. This last equation holds if the generalised continuum hypothesis is assumed.

We formulated as a problem that every good family must contain for some r the finite subgraphs of the r-th shift-graph. We expected that the answer to this question will be negative, but we could not show this. Recently A. Hajnal and P. Komjath [10] showed that the answer is negative. Hajnal conjecture that if F_n, $n = 1, 2, \cdots$ is a good family for all n then there is good family F satisfying $F \supset F_n$, $n = 1, 2, \cdots$. A much stronger (but also much more doubtful) conjecture is that there is a good family F which is almost contained in F_n for every n. Perhaps one should first try to disprove this. The answer is unknown even for the finite subgraphs of the r-th shift-graph.

The intersection of two good families is perhaps always good, but we cannot even exclude the possibility that there are c families of almost disjoint good families of finite graphs. We are, of course, interested only in finite graphs of chromatic number

≥ 4, since our old result with Shelah implies that every G of chromatic number $\geq \aleph_1$ contains all odd circuits for $n > n_0$.

Hajnal and I proved that every graph of chromatic number $\aleph_1$ contains a tree each vertex of which has degree $\aleph_0$, and we also proved that it contains for every n, a $K(n, \aleph_1)$ but it does not have to contain a $K(\aleph_0, \aleph_0)$. Hajnal [9] showed that if $c = \aleph_1$, it does not have to contain a $K(\aleph_0, \aleph_0)$ and a triangle. The problem is open (and is perhaps difficult) whether there is graph of chromatic number $\aleph_1$ which does not contain a $K(\aleph_0; \aleph_0)$ and has no triangle and no pentagon (and in fact no C_{2r+1} for $r \leq K$).

Hajnal and Komjath [10] recently proved the following result of astonishing accuracy: Every G of chromatic number $\aleph_1$ contains a half-graph (i.e. a bipartite graph whose white vertices are $x_1, x_2, \cdots$ and whose black vertices are $y_1, y_2, \cdots$, where x_i is joined to y_j for $j > i$) and another vertex which is joined to all the x_i. On the other hand, if $c = \aleph_1$ is assumed, it does not have to contain two such vertices.

To end this short excursion into transfinite problems, let me state an old problem of Hajnal and myself: Is it true that every G of chromatic number $\aleph_1$ contains a subgraph G' which also has chromatic number $\aleph_1$ and which cannot be disconnected by the omission of a finite number of vertices? We observed that, if true, this is best possible; we gave a simple example of a graph of chromatic number $\aleph_1$ every subgraph of which has vertices of degree $\aleph_0$.

P. Komjath recently proved that every graph G of chromatic number $\aleph_1$ contains for every n, a subgraph G_n of chromatic number $\aleph_1$ which cannot be disconnected by the omission of n vertices and he informed me that he can also insure that there is such a G_n all vertices of which have infinite degree.

As far as I know the following Taylor-like problem has not yet been investigated: Determine the smallest cardinal number m for which if G has chromatic number m, then there is a G' of arbitrarily large chromatic number all of whose denumerable subgraphs are also subgraphs of G. Hajnal observed that it is consistent that every G of chromatic number $\aleph_2$ contains a $K(\aleph_0,\aleph_0)$. Thus it is consistent that $m > \aleph_1$. He suggests that perhaps one can prove (assuming G.C.H.?) that every G of chromatic number $_2$ contains the Hajnal-Komjath graph as a subgraph. Thus the analog of Taylor's conjecture is perhaps $m = \aleph_2$.

Now I discuss some finite problems. El-Zahar and I considered the following problem: Is it true that for every k and ℓ there is an $n(k,\ell)$ so that if the chromatic number of G is $\geq n(k,\ell)$ and G contains no $K(\ell)$, then G contains two vertex-disjoint k-chromatic subgraphs G_1 and G_2 so that there is no edge between G_1 and G_2? We proved this for $k = 3$ and every ℓ, but great difficulties appeared for $k = 4$, and Rödl suggested that the probability method may give a counterexample. It seems to me that this method just fails.

For $k = 3$ the simplest unsolved problem is: Let G be a 5-chromatic graph not containing a $K(4)$. Is it then true that G contains two edges e_1 and e_2 so that the subgraph of G induced by the 4 vertices of e_1 and e_2 only contains these edges? The answer is certainly affirmative if we assume that the chromatic number of G is ≥ 9.

During a recent visit to Israel, Bruce Ruthschild was there and we posed the following problem:

Denote by $G(k;\ell)$ a graph of k vertices and ℓ edges. We say that the pair n, e forces k, ℓ, $(n,e) \to (k,\ell)$, if every $G(n;e)$ contains a $G(k;\ell)$ or a $G(k;\binom{k}{2}-\ell)$ as an induced subgraph. It seems that the most interesting problems arise if $\ell = \frac{1}{2}\binom{k}{2}$. In this case we can of course assume that $c \leq \frac{1}{2}\binom{n}{2}$. We have unfortunately almost no positive results. We observed that if $e > \frac{2n}{3}$ then $(n,e) \to (4,3)$. This clearly does not hold for $e \leq \frac{2n}{3}$. This unfortunately is our only positive result. On the other hand, we observed that if $n > n_o$, then $(n,e) \nrightarrow (5,5)$ for every e (and $n > n_o$). In other words, for every e there is a $\mathcal{G}(n;e)$ which does not contain a $\mathcal{G}(5;5)$ as an induced subgraph and the same holds for a $\mathcal{G}(8;14)$. Graham observed the same method gives that $(n,e) \nrightarrow (12,33)$. We convinced ourselves that for $k > 12$ our method no longer will give a counterexample. The simplest unsolved problem is, unless we overlooked a trivial idea, perhaps interesting and non-trivial: Are there any values of n and e for which $(n,e_n) \rightarrow (9,18)$? Further and determine all these values of n and e_n.

Fan Chung and I spent (wasted?) lots of time on the following problem: Denote by $f(n;k,\ell)$ [1] the smallest integer for which every $\mathcal{G}(n,f(n;k,\ell))$ contains a $\mathcal{G}(k;\ell)$ as a subgraph. Here we of course do not insist that the subgraph should be induced. Also we do not prescribe the structure of our $\mathcal{G}(k;\ell)$. The first interesting and difficult case seems to be: Is it true that

$$(1) \qquad \frac{f(n;\,8,\,13)}{n^{3/2}} \rightarrow \infty \qquad ?$$

We could not prove (1); the probability method seems to fail. Probably $f(n;8,13) > n^{3/2+\varepsilon}$ also holds. It is well known and easy to see that $f(n;8,12) < c\, n^{3/2}$ holds, since every $G(n;c_r\, n^{3/2})$ contains for sufficiently large c_r, a $K(r,2)$, and thus a $K(6,2)$ of 8 vertices and 12 edges. Completely new and interesting questions come up if we also consider the structure of $\mathcal{G}(k;\ell)$, e.g., Simonovits and I [7] proved that every $\mathcal{G}(n;c\, n^{8/5})$ contains a cube - the proof is quite difficult. We believe that our exponent 8/5 is best possible but could not even show that for every c and $n > n_o(c)$ there is a $\mathcal{G}(n;\, c\, n^{3/2})$ which contains no cube as a subgraph. A more general conjecture of Simonovits and myself states that if G is bipartite then the necessary and sufficient conditions of

$$\frac{f(n;\ \mathcal{G})}{n^{3/2}} \to \infty \tag{2}$$

is that $\mathcal{G}$ should have no induced subgraph each vertex of which has degree greater than 2. Perhaps this condition already implies

$$f(n;\ \mathcal{G}) > n^{3/2+\varepsilon}. \tag{3}$$

Conjectures (2) and (3), if true, will probably require some new ideas.

During a recent visit to Calgary, Sauer told me his conjecture: Let C be a sufficiently large constant. Is it true that for every k there is an $f_k(C)$ so that every $G(n;f_k(c)n)$ contains a subgraph each vertex of which has degree $v(x)$, $k < v(x) < Ck$. In other words, the subgraph is quasiregular. Related problems were also stated in our paper with Simonovits and we used the concept of quasiregularity to prove our $\mathcal{G}(n;\ c\ n^{8/5})$ theorem, but as far as I know the conjecture of Sauer is new and is very interesting.

During the 1984 international meeting on graph theory in Kalamazoo, Toft posed the following interesting question: Is there a 4-chromatic edge critical graph of $c_1\ n^2$ edges which can be made bipartite only by the omission of c_2n^2 edges? It is not even known if for every c there is a 4-chromatic critical graph of $c\ \ n_1^2$ edges which can not be made 2-chromatic by the omission of $C\ n$ edges.

Perhaps I might be permitted to make a few historical remarks: A k-chromatic graph is called edge critical if the omission of every edge decreases the chromatic number to $k - 1$. This concept is due to G. Dirac. When I met him in London early in 1949 he told me this definition. I was already at that time interested in extremal problems and immediately asked: What is the largest integer $f(n;k)$ for which there is a $\mathcal{G}(n;\ f(n;k))$ that is k-chromatic and edge critical? In particular, can $f(n;k)$ be greater than $c\,n^2$? To my surprise Dirac showed very soon that for $k \geq 6$, $f(n;k) > c_k\,n^2$ and, in particular $f(n;6) > \frac{n}{4} + cn$. This result has not been improved for more than 35 years, and left the problem open for $k = 4$ and $k = 5$. In 1970 Toft [15] proved that $f(n;4) > \frac{n^2}{16} + cn$. Simonovits and I easily proved that $f(n;4) < \frac{n^2}{4} + cn$. It would be very desirable to determine $f(n;k)$, or, if this is too difficult, to determine

$$\lim \frac{f(n;k)}{n^2} = c_k.$$

The graph of Toft has many vertices of bounded degree. I asked: Is there a 4-chromatic critical graph $G(n)$ each vertex of which has degree $> cn$. (Dirac's 6-chromatic critical graph has this property.) Simonovits [14] and Toft [16] independently found a 4-chromatic critical graph each vertex of which has degree $> cn^{1/3}$. The following question occurred to me: Is there a 4-chromatic critical $G(n;\,c\,n^2)$ which does not contain a very large $K(t,t)$? All examples known to me contain a $K(t,t)$ for $t > c\,n$, but perhaps such an example exists with $t < C \log n$. (Rödl in fact recently constructed such an example).

To end this paper I want to mention some older problems which I find very attractive and which I have perhaps neglected somewhat and which have both a finite and an infinite version. First an old conjecture of Hajnal and myself:

Is it true that for every cardinal number m there is a graph G which contains no $K(4)$ and if one colors the edges of G by m colors there always is a monochromatic triangle. For $m = 2$ this was proved by Folkman and for every $m < \aleph_0$ it was proved by Nesetril and Rödl [11]. For $m \geq \aleph_0$ the problem is open. The strongest and simplest problem which is open is stated as follows (where we assume that the continuum-hypothesis holds): Is it then true that there is a G of power $\aleph_2$ without a K(4) so that if one colors the edges of G by $\aleph_0$ colors there always is a monochromatic triangle. If $c = \aleph_1$ is not assumed, then $\aleph_2$ must be replaced by c^+. I offer a reward of 250 dollars for a proof or disproof (perhaps this offer violates the minimum wage act).

An interesting finite problem remains. For m = 2 Folkman's graph is enormous, it has more than $10^{10^{10^{10^{10^{10^{10^{10}}}}}}}$ vertices and the graph of Nesetril and Rödl is also very large. This made me offer 100 dollars for such a graph of less than 10^{10} vertices (the truth in fact may be very much smaller, there very well could exist such a graph of less than 1000 vertices). Rödl and Szemerédi found such a graph which has perhaps $< 10^{12}$ vertices which does not fall very short of fulfilling my conditions and perhaps can be improved further.

Another old conjecture of Hajnal and myself states that for every k and r there is an f(k,r) so that if G has chromatic number $\geq f(k,r)$, then it contains a subgraph of girth $> k$ and chromatic number $> r$. For $k = 3$ this was answered affirmatively by Rödl [12]. The infinite version of our problem states: Is it true that every graph of chromatic number m contains a subgraph of chromatic number m the smallest odd circuit of which has size $> 2k + 1$? This problem is open even for $k = 1$.

Our triple paper with Hajnal and Szemerédi [6] contains many interesting unsolved finite and infinite problems. Is it true that every graph G of chromatic number $\aleph_1$ contains for every C a finite subgraph G(n) which cannot be made bipartite by the omission of C n edges? Perhaps one can further assume that our G(n) has chromatic number 4. The difficulty again is that so little is known about the critical 4-chromatic graphs.

Let $f(n)$ be a function that tends to infinity as slowly as we please. Is it true that for every k there is a k-chromatic graph so that for each n every subgraph of n vertices of G can be made bipartite by the omission of fewer than $f(n)$ edges. Lovász and Rödl [13] proved this for $f(n) = O(n^{(1/k)-2})$ and Rödl settled the conjecture for triple systems.

Let $F(n)$ tend to infinity as fast as we please. Is there an $\aleph_1$-chromatic G so that for each n every n-chromatic subgraph of G has more than $F(n)$ vertices?

Hajnal, Sauer and I asked in Calgary recently: Let $\mathcal{G}$ be n-chromatic and the smallest odd circuit of which is $2k + 1$. Is it then true that the number of vertices of G is greater than n^{c_k}, where c_k tends to infinity together with k? Perhaps we overlooked a trivial point, but we could not even show that the number of vertices of G must be greater than $n^{2+\varepsilon}$. It seems clear that this must hold if we only assume that G has no triangle and pentagon.

An old problem of mine which has been neglected [2] is stated as follows: Is it true that for every small $\varepsilon > 0$ and infinitely many n there is a regular $\mathcal{G}(n)$ with degree $v(x) = [n^{(1/2)+\varepsilon}]$ so that $\mathcal{G}(n)$ has no triangle and the largest stable set of which has size $v(x)$. I expect that the answer is negative and offer 100 dollars for a proof or disproof.

Here is a final question of mine which I had no time to think over carefully and which might turn out to be trivial. Let $\mathcal{G}(n)$ be a k-chromatic graph. Then clearly $\mathcal{G}(n)$ always has a subgraph of $\le \frac{n+1}{2}$ vertices which has chromatic number $\ge \frac{k+1}{2}$. Can this be strengthened if we assume say that $\mathcal{G}$ has no triangle? (Without some assumption the complete graph shows that the original result is best possible.) As a matter of fact I now believe that no such strengthening is possible. The probability method seems to give that to every $\varepsilon > 0$ there is a $k_0(\varepsilon)$ so that for every $k > k_0(\varepsilon)$ and $n > n_0(\varepsilon,k)$ there is a k-chromatic $G(n)$ of girth ℓ so that every set of εn vertices of which spans a graph of chromatic number $(1+o(1))\varepsilon n$, but I may be wrong since I did not check the details.

References

1. P. Erdös, Extremal problems in graph theory. Theory of Graphs and its Applisations, Proc. Symp. Smolenice (1963) 29-36.
2. P. Erdös, On the construction of certain graphs, J. Combinatorial Theory 1(1966) 149-153.
3. P. Erdös and A. Hajnal, Some remarks on set theory IX. Combinatorial problems in measure theory and set theory. Michigan Math. Journal 11(1964) 107-127. See also [2].
4. P. Erdös and A. Hajnal, On chromatic number of infinite graphs. Theory of Graphs, Proc. Coll. Thinay Hungary Acad. Proc. (1968) 83-98.
5. P. Erdös, A. Hajnal and S. Shelah, On some general properties of chromatic numbers. Topics in Topology, Proc. Coll Kezszthely (1972), Coll. Math. Soc. T. Bolyai Vol. 8 North Holland, Amsterdam (1974) 243-255.
6. P. Erdös, A. Hajnal and E. Szemerédi, On almost bipartite large chromatic graphs. Annals Discrete Math. 12(1982) 117-123.
7. P. Erdös and M. Simonovits, Some extremal problems in graph theory. Combinatorial Theory and its Applications, Proc. Coll. Balatorfured 1969. North Holland, Amsterdam (1970) 377-390.
8. F. Galvin, Chromatic numbers of multigraphs. Periodica Math. Hungarica, 4(1973) 117-119.
9. A. Hajnal, A negative partition relation. Proc. Nat. Acad. Sci. U.S.A. 68(1971) 142-144.

10. A. Hajnal and P. Komjath, What must and what need not be contained in a graph of uncountable chromatic number, Combinatoria 4(1984) 47-52.

11. J. Nesetril and V. Rödl, Ramsey properties of graphs with forbidden complete subgraphs. J. Combinatorial Theory 20B (1976), 243-249.

12. V. Rödl, On the chromatic number of subgraphs of a given graph. Proc. Amer. Math. Soc. 64(1977) 370-371.

13. V. Rödl, Nearly bipartite graphs with large chromatic number. Combinatoria 2(1982) 377-383.

14. M. Simonovits, On colour initial graphs. Studia Sci. Math. Hungar. 7(1972) 67-81.

15. B. Toft, On the maximal number of edges of critical k-chromatic graphs. Studia Sci. Math. Hungar. 5(1970) 469-470.

16. B. Toft, Two theorems on critical 4-chromatic graphs. Studia Sci. Math. Hungar. 7(1972) 83-89.