Some of my old and new problems in elementary number theory and geometry

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I start with an old problem of mine: Denote by $f_k(n)$ the largest integer for which one can find integers $1 \leq a_1 < a_\ell < \ldots < a_t \leq n, t = f_k(n)$ so that no k of the a's should be pairwise relatively prime. My guess was (and is) that one obtains this set by taking the first $k - 1$ primes and the a's are the set of their multiples. Szemerédi remembers that he and Sárkozy proved this if $n > n_o(k)$. I hope they will be able and willing to reconstruct their proof and if possible get rid of the condition $n > n_o(k)$ (perhaps the result no longer holds without this condition). There are two remarks: First of all it is not obvious that the extremal set is obtained by taking the set of all multiples of some set of $k - 1$ primes. If this has been done then one could try to prove that this set is largest if we take the first $k - 1$ primes. This later statement certainly holds for $n > n_o(k)$.

Perhaps in fact the following stronger statement holds: Let $x < a_1 < \ldots < a_t \leq x + n$ and assume that there is no set of k a's which are pairwise relatively prime. Perhaps for every n and x, $\max t = f_k(n)$. The first case which I have not done is: Is it true that among any 23 integers among 30 consecutive integers one can always find 4 which are pairwise relatively prime[2]. This is certainly false for 22 such numbers (take the set of multiples of 2, 3 and 5).

I would like to state one more problem in number theory: One of my oldest theorems (found in 1932) states as follows: [1] Let

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$a_1 < a_2 < \ldots$ be an infinite sequence of integers no one divides the other. Then

$$\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i}$$ converges and in fact there is an absolute constant C for which $\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i} < C$. Probably $\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i}$ is maximal if the a's are the primes. Perhaps a fast computer and a little ingenuity will give a proof. My old problem is quite different and computers will not be of any help. Let $a_1 < a_2 < \ldots$ be any sequence of positive real numbers for which every i, j and k,

(1) $$|ka_i - a_j| \geq 1$$

Observe that if the a's are integers then (1) implies that no a divides any other. Is it true that (1) implies that

$$\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i} < \infty ?$$

I could not even prove that (1) implies that

(2) $$\liminf \left(\frac{1}{X} \sum_{a_i < X} 1 \right) = 0$$

As far as I know the only result in this direction is in the unpublished dissertation of John Haight, he proves that if the a's are rationally independent then (1) in fact implies

(3) $$\liminf \left(\frac{1}{X} \sum_{a_i < X} 1 \right) = 0$$

Besicovitch proved (see Halberstam-Roth Sequences Chapter 5) that (3) does not hold for the integers. I hope Haight will publish his result and will not wait until he has to dedicate it to my memory.

Just one word of caution to the interested reader. Until about 1970 I was quite sure that my conjecture holds, but after a result of Alexander I am no longer so sure. I proved about 50 years ago [2] that if

$1 \le a_1 < a_\ell < \ldots < a_t \le n$ is a sequence of integers for which the products $a_i a_j$ are all distinct then we have ($\pi(n)$ denotes the number of primes $\leq n$),

(4) $$\pi(n) + C_1 \frac{n^{3/4}}{(\log n)^{3/2}} < \max t < \pi(n) + C_2 \frac{n^{3/4}}{(\log n)^{3/2}}$$

where C_1 and C_2 are positive absolute constants. I am sure that in fact there is an absolute constant C for which

(5) $$\max t = (n) + (C + o(1)) \frac{n^{3/4}}{(\log n)^{3/2}}$$

but I was never able to prove (5).

I conjectured that (4) also holds if $1 \le a_1 < \ldots < a_t \le n$ and if we assume that

(6) $$|a_i a_j - a_r a_s| \ge 1$$

holds for every choice of i, j, r, s (the a's are of course not assumed to be integers). I could not even prove that (6) implies

(7) $$t/n \longrightarrow 0.$$

To my great surprise Alexander [3] proved that (6) in fact does not imply (7) and now I am no longer sure that my original conjectures holds. In any case I offer 250 dollars for a proof or disproof and dedicate the problem to my own memory. One final remark. Let $1 < a_1 < a_2 < \dots$ be an infinite sequence of real numbers satisfying (6). Then trivially $\liminf t_n/n = 0$, (where $t_n = \sum_{a_i < n} 1$), perhaps $\lim_{n\to\infty} t_n/n = 0$ also holds.

Now I discuss some problems in geometry. Two of my oldest problems in geometry state: Let $X_1, \dots, X_n$ be any n distinct points in the plane; is it true that they determine at least $cn/\sqrt{\log n}$ distinct distances? The lattice points show that if true then apart from the value of c this conjecture is best possible. I offer 500 dollars for a proof or disproof of this conjecture which seems to me to be very deep [4].

Is it true that the same distance can occur at most $n^{1+c_1/\log\log n}$ times? The lattice points again show that this conjecture if true is best possible. Again I offer 500 dollars for a proof or disproof of this conjecture [4]. These problems can be posed for higher dimensions too but I think the plane seems to be the most difficult and interesting case. A great deal of progress has been made with these problems by Beck, F. Chung, Spencer, Szemeredi and Trotter but the final victory still seems to be very far [5], [6].

Let $X_1, \dots, X_n$ be n points in the plane which determine as few distinct distances as possible. Denote this minimum by $f_2(n)$. Is it then true that the points $X_1, \dots, X_n$ have lattice structure? The first step would be to prove that there are $c\sqrt{n}$ of the $X_i's$ on a line. Is it true that if $X_1, \ldots, X_n$ determine $f_2(n)$ distinct distances then there are four of them which determine only two distances? I cannot even prove that there are four such points which determine only three distances. Is it true that if $X_1, \ldots, X_n$ determine only $o(n)$ distinct distances then there are four of them which determine only three distinct distances. I would guess that the answer is no and I offer 100 dollars for a proof or disproof. Trivially there must be four points which determine at most four distinct distances, since for every X_i there must be k other points (for $n > n(k)$) which are on a circle of center X_i . Suppose we assume that no such circle exists and that no k' of the points are on a line. How many distinct distances must our points determine? Furedi and I considered this question some time ago. It would even be of interest to find n points no three on a line no four on a circle which determine $o(n^2)$ distances.

Many more problems of this type are posed in our joint papers with G. Purdy.

Croft, Purdy and I conjectured that if there are n points in the plane, then the number of lines which contain $\geq k$ of the points is $< \frac{cn^2}{k^3}$, this was proved by Szemeredi and Trotter and in a weaker form by J. Beck. The paper of Szemerédi and Trotter contains many other deep and interesting geometry problems, but I have to refer to their paper for these [6]. Here I mention only one problem: Let there be given n points in the plane, their result implies that the number of lines which contain $\geq \sqrt{n}$ of the points is $< c\sqrt{n}$. It is easy to give n points which determine $2\sqrt{n} + 2$ lines which contain $\sqrt{n}$ of the points, the rectangular lattice shows this. Sah (unpublished) showed that one can find $3\sqrt{n}$ such lines and it is perhaps not hopeless to determine exactly (or at least asymptotically) the maximum number of distinct lines which pass through $\geq \sqrt{n}$ of our n points.

To end this paper I discuss some Euclidean Ramsey problems. We called a finite subset S of some Euclidean space Ramsey [7] if for every k there is an $n_k = n(S,k)$ so that if we partition E_{n_k} (i.e. the n_k dimensional Euclidean space) into k subsets $\{A_i\}$, $1 \leq i \leq k$ in an arbitrary way, at least one of these subsets say A_i, contains a subset which is congruent to S. We proved that all parallelepipeds are Ramsey and that all sets which are Ramsey are spherical. The simplest unsolved problems were: are all obtuse angled triangles Ramsey? and is the regular pentagon Ramsey?

P. Frankl and V. Rödl recently proved that all simplices are Ramsey, they prove many other deep and interesting results, but many unsolved problems remain, e.g. perhaps every finite set which is spherical is Ramsey. Another interesting open problem is: Let (a, b, c) be any non equilateral triangle. Is it true that if one partitions the plane into two parts $S_1 \cup S_2$ then for some $i = 1$ or 2, S_i contains three points (x, y, z) so that the triangle (x, y, z) is congruent to (a, b, c) ?

Another interesting problem states as follows: Let S be a subset of the plane so that no two points of S have distance

1. We conjectured that $\bar{S}$ (the complement of S) then contains the vertices of a unit square. R. Juhasz [8] proved this and in fact she proved that $\bar{S}$ contains to every four points a set congruent to it. She showed that this no longer holds for 12 points but is it true for 5 points? And in fact does $\bar{S}$ contain the vertices of a regular pentagon?

Five final problems: Let there be given n points in the plane no three on a line and no four on a circle. Can it happen that one of the distances occurs $n - 1$ times, one $n - 2$ times, etc. I. Palasta showed [9] that this is possible for $n = 7$. I do not think it is possible for large n in fact if the n points are in general position (i.e. no three on a line and no four on a circle) and n is large. They probably determine more than n distinct distances, but I know nothing about this.

Let there be given n points in the plane no five on a line. Is it true that they determine at most $o(n^2)$ lines which go through 4 of the points.
B. Grunbaum [10] showed that one can have $cn^{3/2}$ such lines and this very well could be best possible.

Let $X_1, \ldots, X_n$ be n points on a line. Denote by $f(n)$ the maximum number of distinct unit circles which pass through at least three of our points. I conjectured
$f(n)/n \to \infty$, $f(n)/n^2 \to 0$. Elekes [11] found a very ingenious construction for $f(n) > c_1 n^{3/2}$, perhaps in fact
$f(n) < c_2 n^{3/3}$.

An old problem of L. Moser and myself states that if

$X_1, \ldots, X_n$ is a convex n-gon then no distance can occur more than cn times, in fact perhaps $\frac{5n}{3}$ is best possible.

It is very annoying that we got nowhere with this very elementary problem.

Finally let $h(n)$ be the largest integer so that among any n distinct points in the plane one can always find $h(n)$ of them so that no two of these $h(n)$ points are at distance 1.

Determine if possible $h(n)$ and if this is impossible try to determine $\lim h(n)/n$. Perhaps $h(n) > \frac{n}{4}$. More generally let $h(n; R_1, \ldots, R_r)$ be the largest integer so that among any n distinct points in the plane one can always find $h(n; R_1, \ldots, R_r)$ of them so that no two of these $h(n; R_1, \ldots, R_r)$ points are at distance $R_1, \ldots, R_r$.

$h_r(n) = \min_{R_1, \ldots, R_r} h(n; R_1, \ldots, R_r)$. Determine or estimate $h_r(n)$ as well as possible.

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