# Some Solved and Unsolved Problems of Mine in Number Theory

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I.

Let $p_1 < p_2 < \ldots$ be the sequence of consecutive primes. Put $p_{n+1} - p_n = d_n$. It is well known that the sequence $d_n$ behaves very irregularly. An old result of Ricci and myself states that the set of limit points of $d_n/\log n$ has positive measure. "It is of course clear to every right thinking person" that the set $d_n/\log n$ is dense in $(0, \infty)$, but we are very far from being able to prove this ("we" here stands for the collective intelligence (or rather stupidity) of the human race). By a result of Westzynthius, $\infty$ is a limit point of $d_n/\log n$ but no other limit point is known.

Turan and I easily proved that both inequalities $d_n > d_{n+1}$ and $d_n < d_{n+1}$ have infinitely many solutions; we noticed with some annoyance and disappointment that we could not prove that $d_n > d_{n+1} > d_{n+2}$ has infinitely many solutions. We were even more annoyed when we soon noticed that we could not even prove that either $d_n > d_{n+1} > d_{n+2}$ or $d_n < d_{n+1} < d_{n+2}$ has infinitely many solutions. In other words we could not prove that for every $n$, $(-1)^i(d_{n+i+1} - d_{n+i})$ must have infinitely many changes of sign. I offer 250 dollars for a proof of this conjecture (and 25,000 for a disproof).

Put $f(n) = \log n \log\log n \log\log\log\log n/(\log\log\log n)^2$. Rankin proved in 1938 that

$$\text{(1)} \qquad \limsup \left(\frac{d_n}{f(n)}\right) > 0 .$$

I offered 10,000 dollars for a proof of

$$\text{(2)} \qquad \limsup \left(\frac{d_n}{f(n)}\right) = \infty .$$

Thirty years ago I proved that

$$\text{(3)} \qquad \limsup \left( \frac{\min(d_n, d_{n+1})}{f(n)} \right) > 0 ,$$

and conjectured that for every $k$

$$\text{(4)} \qquad \limsup \left( \frac{\min(d_n, \ldots, d_{n+k})}{f(n)} \right) > 0 .$$

I could not even prove (4) for $k = 2$. Recently in a brilliant paper Maier proved (4) for every $k$.

I proved

$$\text{(5)} \qquad \liminf d_n/\log n < 1$$

and could not even prove

$$\text{(6)} \qquad \liminf \max(d_n, d_{n+1})/\log n < 1 .$$

I offer 500 dollars for a proof of

$$\liminf \max(d_n, d_{n+1}, \ldots, d_{n+k})/\log n < 1$$

and 250 dollars for the proof of (6).

An old conjecture on primes states that there are arbitrarily long arithmetic progressions among the primes. The longest known progression has 17 terms and is due to Weintraub. I conjectured more than 40 years ago that if $a_1 < a_2 < \ldots$ is a sequence of integers for which $\sum_{i=1}^{\infty} \frac{1}{a_i} = \infty$ then the $a_i$'s contain arbitrarily long arithmetic progressions. I offer 3,000 dollars for a proof or disproof. No doubt the following very much stronger result holds: For every $k$ there is an $n$ for which $d_n = \ldots = d_{n+k}$. It is not even known that $d_n = d_{n+1}$ has infinitely many solutions. Rényi and I proved 35 years ago that the density of integers $n$ for which $d_n = d_{n+1}$ is $0$, and in fact our proof gives without much difficulty that the density of integers $n$ for which the $k$ numbers $d_n, \ldots, d_{n+k-1}$ are all distinct is one. Let us henceforth consider only those $n$ for which this happens. Order the $k$ integers $d_n, \ldots, d_{n+k-1}$ by size. This gives a permutation $\{i_1, \ldots, i_k\}$ of $1, \ldots, k$. No doubt all the $k!$ permutations occur infinitely often, and in fact if there is justice in heaven or earth each occurs with the same frequence $\frac{1}{k!}$. (There clearly is no justice in heaven or earth, but the conjecture nevertheless holds since there certainly is justice in Mathematics. Unfortunately I cannot at present give a better proof for the conjecture.)

Denote by $F(k)$ the number of permutations which must occur among the $i_1, \ldots, i_k$. $F(2) = 2$ is the simple result of Turán and myself, and perhaps it will not be too difficult to prove $F(3) \geqslant 4$. The following problem which just occurred to me might be of some interest here. Let $a_1 < a_2 < \ldots$ be a sequence of integers for which $a_n/n \log n \to 1$. Put $a_{n+1} - a_n = D_n$ and assume that for every $k$ and almost all $n$ the integers $D_n, \ldots, D_{n+k-1}$ are all distinct. Order the integers $D_n, \ldots, D_{n+k-1}$ by size; this gives a permutation $\{i_1, \ldots, i_k\}$ and denote by $G(k)$ the smallest possible number of these permutations. Again it is easy to see that $G(2) = 2$ and perhaps $G(k)$ can be determined exactly. (The problem is to be understood as follows: We assume only $a_n/n \log n \to 1$ and that for almost all $n$ the $k$ integers $D_n, \ldots, D_{n+k-1}$ are all distinct.) Can one show $F(k) > G(k)$ for some $k$?

One final problem. De Bruijn, Turán, Katai and I independently investigated the function

$$f(n) = \sum_{p < n} \frac{1}{n-p},$$

hoping to find something new about the primes. We of course noticed that

(7) $$\frac{1}{x}\sum_{n=1}^{x} f(n) \to 1 \quad\text{and}\quad f(n) > c \quad\text{for all } n .$$

The first equation of (7) follows from the prime number theorem and the second from the theorem of Hoheisel. Also $\frac{1}{x}\sum_{n<x} f^2(n) < C$ easily follows from Bruns' method. I once claimed that it follows from Bruns' method that

(8) $$\sum_{n=1}^{x} f^2(n) = x + o(x) .$$

Pomerance noted that I am wrong and (8) (which as far as I know is still open) almost certainly needs more new ideas. Besides the proof of (8) the interesting and difficult questions are:

(9) $$\limsup f(n) = \infty, \quad f(n)/\log\log n \to 0 .$$

Probably

(10) $$\liminf f(n) = 1 ,$$

and perhaps

(11) $$c_1 < \limsup f(n)/\log\log\log n < c_2 , \quad 0 < c_1 \leq c_2 < \infty .$$

I am most doubtful about (11).

I recently conjectured that if $a_1 < a_2 < \dots$ satisfies $a_n/n \log n \to 1$ and we put $F(n) = \sum_{a_i < n} \frac{1}{n-a_i}$, then 1 is always a limit point of the sequence $F(n)$.

Montgomery found an ingenious and highly nontrivial proof of this conjecture. Ruzsa and I further conjectured that 2 is also a limit point of the sequence $F(n)$. It is a simple exercise to show that no other $\alpha$, $0 \leq \alpha \leq \infty$ is a compulsory limit point of this sequence.

II.

Now about divisor problems. Denote by $\tau(n)$ the number of divisors of $n$; $1 = d_1 < d_2 < \ldots < d_{\tau(n)} = n$ are the consecutive divisors of $n$. I conjectured more than 40 years ago that the density of integers for which

$$\min d_{i+1}/d_i < 2 \tag{12}$$

is 1. More than 30 years ago I proved that the density of the integers satisfying (12) exists but I could never prove that it is 1. At this moment the question is still open and in fact all the recent results of Tenenbaum and myself seem to be consistent with the assumption that the conjecture is false, but nevertheless we both believe it to be true.

Denote by $\tau^*(n)$ the number of integers $k$ such that $2^k < d_i < 2^{k+1}$ for some $i$. I conjectured that $\tau^*(n)/\tau(n) \to 0$ for almost all $n$; this would imply conjecture (12). Tenenbaum and I proved that this conjecture is completely wrongheaded and that in fact the density of integers for which $\tau^*(n)/\tau(n) \to 0$ is $0$.

Denote by $g(n)$ the number of indices $i$ for which $d_i | d_{i+1}$. At the meeting in Durham in 1979 Montgomery conjectured that for almost all integers $g(n) \neq O(\tau(n))$. I offered Montgomery a bet 10 to 1 that he was wrong. But it turned out that my intuition misled me, and in a forthcoming paper Tenenbaum and I prove Montgomery's conjecture. We also prove that

$$h(n) = \frac{1}{\tau(n)} \sum_{i=1}^{\tau(n)-1} \frac{d_i}{d_{i+1}}$$

has a distribution function. We could not prove that the distribution function is continuous.

Hooley investigated and used the following function:

$$\Delta(n) = \max_t \sum_{t \leqslant d_i \leqslant 2t} 1 .$$

Hooley proved

$$\sum_{n=1}^{x} \Delta(n) < c_0 x(\log x)^c \tag{13}$$

where $c$ is a small positive constant. I proved that

$$\frac{1}{x} \sum_{n=1}^{x} \Delta(n) \to \infty . \tag{14}$$

Hall and Tenenbaum improved the value of $c$ and also proved

$$\sum_{n=1}^{x} \Delta(n) > c_1 x \log \log x . \tag{15}$$

Hooley thought that perhaps for every $\varepsilon > 0$ and $x > x_0(\varepsilon)$,

$$\sum_{n=1}^{x} \Delta(n) < x(\log x)^{\varepsilon} \tag{16}$$

holds. (Note that in the introduction to the paper referred to below, Hooley has qualified this suggestion.) It would be of interest to estimate $\sum_{n=1}^{x} \Delta(n)$ as accurately as possible. It seems likely that a proof of my conjecture $\Delta(n) \geqslant 2$ for almost all $n$ also will give that $\Delta(n) \to \infty$ if we neglect a sequence of density $0$.

Put

$$\Delta_c(n) = \max_t \sum_{t \leqslant d_i < ct} 1$$

(so that $\Delta(n) = \Delta_2(n)$). Perhaps $\Delta_c(n)/\Delta_2(n) \to 1$ holds for almost all $n$ and every $c$, $1 < c < \infty$. Also perhaps for every $c$, $1 < c < \infty$,

$$\sum_{n=1}^{x} \Delta_c(n) / \sum_{n=1}^{x} \Delta_2(n) \to 1 .$$

The following conjecture (the author of which I cannot place) states: Put ($\mu(n)$ is the well known function of Mobius)

$$r(n) = \max_t \left| \sum_{1 \leqslant d_i < t} \mu(d_i) \right| .$$

Is it true that for almost all $n$, $r(n) \to \infty$?

R.R. Hall and I investigated

$$f(n) = \sum_{(d_i, d_{i+1})=1} 1 . \tag{17}$$

We proved that for infinitely many $n$

$$f(n) > \exp(\log \log n)^{2-\varepsilon} . \tag{18}$$

During the meeting at Lyon on ordered sets Tenenbaum visited me for a day and we did some "illegal thinking" about $f(n)$. "Illegal thinking" is an important concept introduced by R.L. Graham and myself. It describes the situation when a mathematician works on a problem when he is really supposed to work on another one.

We improved (18) a great deal. Let $n_k = \prod_{j=1}^{k} p_j$ be the product of the first $k$ primes. The prime number theorem implies

$$n_k = \exp\{(1 + o(1))k \log k\} . \tag{19}$$

Since $\tau(n_k) = 2^k$ we obtain by a simple computation that for at least $2^{k-1}$ indices $i$

$$d_{i+1}/d_i < 1 + \frac{10k \log k}{2^k} . \tag{20}$$

Henceforth we will only consider the $d_i$ satisfying (20). The consecutive divisors satisfying (20) unfortunately do not have to be relatively prime. To overcome this difficulty we construct two consecutive divisors $d_i/T$, $d_x$ satisfying

$$(21) \qquad (d_i/T\,,\,d_x) = 1 \quad \text{and} \quad Td_x/d_i \leq d_{i+1}/d_i$$

by the following simple process. First of all consider

$$d_i/(d_i\,,\,d_{i+1})\ ,\quad d_{i+1}/(d_i\,,\,d_{i+1})\ .$$

These two divisors are certainly relatively prime, but they do not have to be consecutive. Let $d_y$ be the smallest divisor of $n_k$ which is greater than $d_i/(d_i\,,\,d_{i+1})$. These two consecutive divisors satisfy the second condition of (21). If the first is not satisfied we repeat the same process with $d_i/(d_i\,,\,d_{i+1})$ and $d_y$ and continue. Clearly in a finite number of steps we arrive at two consecutive relatively prime divisors $d_z\,,\,d_{z+1}$ which satisfy (21).

The only trouble now is that the same pair $d_z\,,\,d_{z+1}$ may originate from many of the consecutive pairs $d_\ell\,,\,d_{\ell+1}$. Observe that by (20) and (21) we have

$$(22) \qquad d_{z+1}/d_z < 1 + \frac{10k \log k}{2^k}\ .$$

From (22) and $d_{z+1}/d_z \geq 1 + \frac{1}{d_z}$ we immediately obtain

$$(23) \qquad d_z \geq 2^k/10k \log k\ .$$

By the prime number theorem all prime factors of $n_k$ -- and therefore of $d_z$ -- are less than $2k \log k$ and thus by (23) ($v(m)$ denotes the number of distinct prime factors of $m$)

$$(24) \qquad v(d_z) > (1 + o(1))(k/\log k)\ .$$

Finally notice that if $d_z\,,\,d_{z+1}$ originated by our process from $d_i\,,\,d_{i+1}$ we must have $d_z | d_i$. Thus the number of choices of $d_i$ is at most

$$(25) \qquad 2^{k-v(d_z)} \geq 2^{k-(1+o(1))k/\log k}\ .$$

We remind the reader that the number of possible choices of the pair $d_i, d_{i+1}$ satisfying (20) was at least $2^{k-1}$. Thus the number of distinct indices $(d_z, d_{z+1}) = 1$ is by (25) greater than $(\log 2)\exp\{((\log 2) + o(1))\left(\frac{k}{\log k}\right)\}$. Thus finally from (19) we have

$$(26) \qquad \begin{aligned} f(n_k) &\geqslant (\log 2)\exp\{((\log 2) + o(1))\left(\frac{k}{\log k}\right)\} \\ &\geqslant (\log 2)\exp\{((\log 2) + o(1))\left(\frac{\log n_k}{(\log\log n_k)^2}\right)\} . \end{aligned}$$

I hope the reader will agree that our "illegal thinking" was not a complete waste of time.

How close is (26) to being best possible? Clearly

$$f(n) \leqslant \tau(n) \leqslant \exp\{((\log 2) + o(1))\left(\frac{\log n}{\log\log n}\right)\} .$$

Is it true that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$,

$$(27) \qquad f(n) < \exp(\varepsilon \log n/\log\log n) \ ?$$

At present we cannot decide this question. We also tried to prove that

$$\frac{1}{x \log\log x} \sum_{n=1}^{x} f(n) \to \infty .$$

Here we failed completely. All we could show is that

$$\liminf \frac{1}{x \log\log x} \sum_{n=1}^{x} f(n) > 1 + c$$

for some $1 < c < 2$, which is very unsatisfactory.

Put now $f(n) = f_2(n)$. I tried to estimate $f_3(n)$, the number of indices $i$ for which $d_i, d_{i+1}, d_{i+2}$ are pairwise relatively prime. I never could get a nontrivial result. $f_3(n) > c(\log n/\log\log n)$ holds for infinitely many $n$, but the proof is essentially trivial and I never could get anything better. (Early in August 1982 Sárközy and I proved that for every $r$ and $c$ there is an $n$ for which $f_r(n) > (\log n)$ .)

Put

$$h(n) = \sum_{(d_i, d_{i+1})=1} (d_i/d_{i+1}) .$$

Is it true that for almost all $n$, $h(n) \to \infty$? Can one get an asymptotic formula -- or a good inequality -- for $\sum_{n=1}^{x} h(n)$ ?

During my talk at Austin (June 1982) I announced the following fairly recent conjecture of mine. Define

$$(28) \qquad L_2(n) = \sum_{i=1}^{\tau(n)-1} (d_{i+1}/d_i - 1)^2 .$$

The conjecture states that $L_2(n)$ (and more generally $L_{1+\varepsilon}(n)$) is bounded for infinitely many $n$. More specifically I conjectured that $L_2(n!)$ is bounded.

Dr. M. Vose proved my first conjecture for $L_{1+\varepsilon}(n)$ in a very ingenious way. His method does not seem to give the boundedness of $L_2(n!)$: This was recently proved by Tenenbaum.

One final remark. A little elementary analysis gives that the boundedness of (28) implies

$$(29) \qquad \tau(n) > c(\log n)^2 .$$

I expect that (28) implies

$$\tau(n)/(\log n)^2 \to \infty$$

and, in fact, it is quite possible that (28) implies a much sharper lower bound for $\tau(n)$.

III.

Two conjectures from the last century on consecutive integers have been settled in the last decade. Catalan conjectured that 8 and 9 are the only consecutive powers. Tijdeman proved that there is an absolute constant c which can be explicitly determined so that if there are two consecutive powers they must be less than c.

In the first half of the last century it was conjectured that the product of consecutive integers is never a power. After many preliminary results Selfridge and I proved this conjecture. In fact we proved that for every $\ell$ there is a $p > k$ and $\alpha \not\equiv 0 \pmod{\ell}$ for which $p^{\alpha} \| \prod_{i=1}^{k} (n+i)$. We conjectured that for $k > 3$ there is a $p > k$ for which $p \| \prod_{i=1}^{k} (n+i)$. If true, this seems very deep.

Put ($p(m)$ is the least, $P(m)$ the largest prime factor of $m$)

$$n + i = a_i^{(n)} b_i^{(n)} \quad \text{where} \quad P(a_i^{(n)}) \leq k \,, \quad p(b_i^{(n)}) > k \,.$$

An old conjecture of mine states that

$$\frac{1}{k} \min_{1 \leq i \leq k} a_i^{(n)} \to 0$$

if $k$ tends to infinity. In other words to every $\varepsilon$ and $k > k_0$, we have, for every $n$,

$$\min_{1 \leq i \leq k} a_i^{(n)} < \varepsilon k \,. \tag{30}$$

A simple averaging process gives

$$\min_{1 \leq i \leq k} a_i^{(n)} < Ck$$

for an absolute constant $C$. I made no progress with the proof of (30). This seems to me to be an attractive conjecture, and I offer 100 dollars for a proof or disproof.

Gordon and I investigated the question of how many consecutive values of $a_i^{(n)}$, $i = 1, 2, \ldots$ can be distinct. Denote by $f(k)$ the largest integer for which there is an integer $n$ so that all the values $a_i^{(n)}$, $1 \leq i \leq f(k)$ are distinct. We proved $f(k) < (2+o(1))k$ and conjectured that $f(k) < (1+o(1))k$. I offer 100 dollars for the proof or disproof of this attractive conjecture. A related problem states: Denote by $h(k)$ the largest integer so that there are at least $h(k)$ distinct numbers among the $a_i^{(n)}$, $1 \leq i \leq k$. Perhaps there is a constant $c$ so that $h(k) > ck$. I have no real evidence for this conjecture, which very well could be wrong.

Ruzsa and I made some little progress (following an observation of Ruzsa). In (30) $\varepsilon k$ cannot be replaced by $(\varepsilon k/\log)$ and $f(k) = k + o(k)$ cannot be replaced by $k + O\left(\frac{k}{\log k}\right)$. I once conjectured that

$$\min_n \sum_{i=1}^{k} \frac{1}{a_i^{(n)}} = h(k)$$

tends to infinity, but our observation shows that $h(k) > c \log \log k$, if true, is certainly best possible. We hope to return to these questions later.

I once conjectured that for every $n \geq 2k$, there is an $i$, $0 \leq i < k$, for which $(n-i) \mid \binom{n}{k}$. Schinzel and I proved that this conjecture fails for infinitely many $n$. Perhaps there is a $c > 0$ so that there is an $m$, $cn < m \leq n$, with $m \mid \binom{n}{k}$.

Schinzel conjectured that for every $k$ there are infinitely many integers $n$ for which $v\left(\binom{n}{k}\right) = k$ ($v(n)$ denotes the number of distinct prime factors of $n$). The conjecture is of the same depth as the prime k-tuple conjecture of Hardy and Littlewood and is probably hopeless already for $k = 2$.

It is easy to see that for all but a finite number of values of $n$, $\binom{n}{k}$ has at least $k-1$ distinct prime factors greater than $k$. Is it true that for every $k$ there are infinitely many values of $n$ for which $v(\binom{n}{k}) = k$ and one of these prime factors is $\leqslant k$? This conjecture is even much more hopeless than that of Schinzel. It almost certainly holds for $k = 2$ but is very doubtful for $k > 2$. In fact denote by $p_i(n, k)$ the i-th largest prime factor of $\binom{n}{k}$. It is not difficult to show that for fixed $k$ and $n \to \infty$, $p_{k-1}(n, k) \to \infty$. I cannot decide if $p_k(n, k) \to \infty$. This is probably false for small values of $k$.

Selfridge and I investigated the integers $\binom{n}{k}$ for which

$$\binom{n}{k} = p_1 p_2 \cdots p_{k-r} , \quad k < p_1 < \cdots < p_{k-r} .$$

We define $r$ as the deficiency of $\binom{n}{k}$. $\binom{47}{11}$ has deficiency 4 and perhaps this is the largest possible deficiency. Observe that perhaps there are only a finite number of integers $n$ and $k$ with positive deficiency. On the other hand, as far as we know, there may be integers $k$ and $n$ whose deficiency is greater than $k(1-\varepsilon)$, though this seems very unlikely.

R. Graham and I observed that for every $n \geqslant 2k$,

$$(31) \qquad v(\binom{n}{k}) \geqslant (1+o(1)) \left(\frac{k}{\log k}\right) \log 4 ,$$

and that $\log 4$ is best possible. The value of $n$ for which the minimum in (31) is assumed is $(1+o(1))2k$, but we could not prove that for infinitely many $k$ the minimum is really assumed for $n = 2k$.

I conjectured that for every $\varepsilon > 0$ there is a $k_0$ so that for every $k > k_0$ and any set of $k$ consecutive integers $n+1, \ldots, n+k$ there always is at least one, say

$n+i$, $1 \leqslant i \leqslant k$, which has no divisor $d$ satisfying $\varepsilon k \leqslant d \leqslant k$. Ruzsa observed that the conjecture becomes false if $\varepsilon k$ is being replaced by $\frac{ck}{\log k}$.

Is it true that there is an absolute constant $c$ so that the number of integers $m+i$, $1 \leqslant i \leqslant n$ which have a prime factor $p$, $\frac{n}{3} < p < \frac{n}{2}$ is greater than $c(n/\log n)$? I could get nowhere with this easy looking problem and perhaps I overlooked a trivial argument. Observe that for suitable $m$ there may be only one $i$ so that $m+i$ $(1 \leqslant i \leqslant n)$ has a prime factor $p$ satisfying $\frac{n}{2} < p \leqslant n$. To see this choose $m$ so that $m + [\frac{n}{2}]$ is a multiple of all the primes $p$, $\frac{n}{2} < p \leqslant n$.

To end the paper I state a few more problems.

Is it true that there are infinitely many integers $n$ for which for every $1 \leqslant k < n$ ($[a, b]$ is the least common multiple of $a$ and $b$)

$$(32) \qquad [n+1, \ldots, n+k] > [n-1, \ldots, n-k] \ ?$$

I expect that the answer is affirmative, but that the density of the integers $n$ satisfying (32) is $0$. Early in 1983 Straus and I proved the second conjecture.

Put

$$\binom{n}{k} = u(n;k)v(n;k)$$

where $P(u(n;k)) \leqslant k$ and $p(v(n;k)) > k$. A well known theorem of Mahler states that for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon, k)$ so that for every $n > n_0$

$$(33) \qquad u(n;k) < n^{1+\varepsilon} .$$

(33) is a very pretty and useful inequality; the only trouble is that it is not effective. An effective inequality

replacing (33) would be useful. It is easy to see that for every $\eta$ and $k > k_0(\eta)$ there are infinitely many values of n for which

$$(34) \qquad u(n;k) > cn \log n \cdot e^{k(1-\eta)} .$$

Perhaps the following inequality holds: There is an absolute constant C so that for every n and k $(n \geq 2k)$

$$(35) \qquad u(n;k) < Cn^2 e^{2k} .$$

In view of (34), (35) cannot be too far from being best possible, though perhaps $n^2 e^{2k}$ can be replaced by $n(\log n)^{c_1} e^{k(1+\eta)}$. Perhaps for applications it would be more important to determine the hypothetical constant C explicitly. I could not disprove that $(U(n;k) = \max_{m \leq n} u(m;k))$

$$(36) \qquad U(n;k) < C_k n \log n ,$$

but perhaps (36) is too optimistic; perhaps (33) holds with $(\log n)^{c_k}$ instead of $\log n$. I could not decide whether

$$(37) \qquad \sum_{n=2k}^{\infty} \frac{1}{U(n;k)}$$

diverges. Perhaps this will not be difficult.

Let $k = k(n)$ be the largest integer for which $p(\binom{n}{k}) > k$. I can show that for infinitely many n, $k > \frac{c \log n}{\log \log n}$ but I have no nontrivial upper bound for k.

Denote by $A(u,v)$ the product of those primes p for which $p \parallel \prod_{i=1}^{v-u} (u+i)$ ($p \parallel d$ means $p | d$, $p^2 \nmid d$). Is it true that for almost all squarefree numbers n and all u, v with $u < n < v$, $A(u,v) \geq n$?

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