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PROBLEMS AND RESULTS ON ADDITIVE PROPERTIES OF GENERAL SEQUENCES. II

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1. Let $\mathcal{A}=\{a_1, a_2, \dots\}$ $(a_1<a_2<\dots)$ be an infinite sequence of positive integers. Denote by $R(n)$ the number of solutions of $a_i+a_j=n$. Starting from a problem of Sidon, P. Erdős [1] proved the following theorem (by using probabilistic methods): There is a sequence $\mathcal{A}$ so that there are two constants c_1 and c_2 for which for every n

$$c_1 \log n < R(n) < c_2 \log n. \tag{1}$$

On the other hand, an old conjecture of Erdős states that for no sequence $\mathcal{A}$ can we have

$$\frac{R(n)}{\log n} \to c \quad (0<c<+\infty). \tag{2}$$

(See [2] and [4] for further related results and problems.)

These problems led us to study the question: how regular can be the behaviour of the function $R(n)$? In part I [3] of this paper, we proved the following results:

THEOREM 1. *If $F(n)$ is an arithmetic function such that*

$$F(n) \to +\infty, \tag{3}$$

$$F(n+1) \geqq F(n) \quad for \quad n \geqq n_0 \tag{4}$$

and

$$F(n) = o\left(\frac{n}{(\log n)^2}\right), \tag{5}$$

and we write

$$\Delta(N) = \sum_{n=1}^{N} \big(R(n)-F(n)\big)^2,$$

then $\Delta(N)=o\big(NF(N)\big)$ cannot hold.

COROLLARY 1. *If $F(n)$ is an arithmetic function satisfying* (3), (4) *and* (5), *then*

$$\max_{n \leqq N} |R(n)-F(n)| = o\big((F(n))^{1/2}\big) \tag{6}$$

cannot hold.

(In fact, Theorem 1 says that (6) is impossible in square mean.)

The aim of this paper is to show that the above results are nearly best possible. We will prove the following theorem:

THEOREM 2. *If $F(n)$ is an arithmetic function satisfying*

$$F(n) > 36 \log n \quad for \quad n > n_0, \tag{7}$$

and there exist a real function $g(x)$, defined for $0<x<+\infty$, and real numbers x_0, n_1 such that

(i) *$g'(x)$ exists and it is continuous for $0<x<+\infty$,*

(ii) *$g'(x) \leqq 0$ for $x \geqq x_0$,*

(iii) *$0<g(x)<1$ for $x \geqq x_0$,*

(iv) *$\left|F(n)-2\int_0^{n/2} g(x)g(n-x)\,dx\right| < (F(n)\log n)^{1/2}$ for $n>n_1$,*

then there exists a sequence $\mathcal{A}$ such that

$$|R(n)-F(n)| < 8\big(F(n)\log n\big)^{1/2} \quad for \quad n > n_2.$$

By choosing $F(n)$ and $g(x)$ in Theorem 2 in an appropriate way, the following corollaries can be derived from Theorem 2:

COROLLARY 2. *If β is an arbitrary real number such that $\beta > 8/\pi^{1/2}$, then there exists an infinite sequence $\mathcal{A}$ such that* (1) *holds with $(0<)c_1=\beta^2\pi-8\beta\pi^{1/2}$, $c_2=\beta^2\pi+8\beta\pi^{1/2}$.*

(So that, e.g., choosing $\beta=5$, we obtain that (1) holds with $c_1=6<\beta^2\pi-8\beta\pi^{1/2}$ and $c_2=151>\beta^2\pi+8\beta\pi^{1/2}$.)

COROLLARY 3. *If $G(x)$ is a real function defined in $(0, +\infty)$ and such that*

(i) $\lim_{x\to+\infty} \dfrac{G(x)}{\log x} = +\infty$,

(ii) $G(x)=o(x)$,

(iii) *$G'(x)$ exists and it is continuous for $0<x<+\infty$,*

(iv) *$G'(x)>0$ for $x>x_0$*

and

(v) $G'(x) = o\left(\dfrac{G(x)}{x}\right)$,

then there exists a sequence $\mathcal{A}$ such that

$$\lim_{n\to+\infty} \frac{R(n)}{G(n)} = 1.$$

(So that, e.g., there exists a sequence $\mathcal{A}$ with $R(n) \sim \log n \log\log n$.)

COROLLARY 4. *If $0<\alpha<1$, then there exists a sequence $\mathcal{A}$ such that*

$$|R(n)-n^\alpha| < 8n^{\alpha/2}(\log n)^{1/2} \quad for \quad n > n_0.$$

In fact, in order to derive Corollaries 2, 3 and 4 from Theorem 2, we have to use Theorem 2 with $\beta\left(\dfrac{\log x}{x}\right)^{1/2}, \left(\dfrac{G(x)}{\pi x}\right)^{1/2}$, $cx^{(\alpha-1)/2}$ (where $c=c(\alpha)$) and $\beta^2\pi\log n$, $2\int_1^{n/2} g(x)g(n-x)\,dx$, n^α in place of $g(x)$ and $F(n)$, respectively.

2. Sections 2, 3 and 4 will be devoted to the proof of Theorem 2. The proof is based on the probabilistic method of Erdős and Rényi [1], [2]. The Halberstam—Roth book [4] contains an excellent exposition of this method thus we use the terminology and notation of this book. In this section, we give a survey of those notations, facts and results connected with this probabilistic method which will be needed in the proof of Theorem 2.

Let Ω denote the set of the strictly increasing sequences of positive integers.

LEMMA 1. *Let*

$$\alpha_1, \alpha_2, \alpha_3, \ldots \tag{8}$$

be real numbers satisfying

$$0 < \alpha_n < 1 \quad (n = 1, 2, \ldots). \tag{9}$$

Then there exists a probability space (Ω, S, μ) with the following two properties:

(i) *For every natural number n, the event $B^{(n)} = \{\omega\colon \omega \in \Omega,\ n \in \omega\}$ is measurable, and $\mu(B^{(n)}) = \alpha_n$.*

(ii) *The events $B^{(1)}, B^{(2)}, \ldots$ are independent.*

This lemma is identical with Theorem 13 in [4], p. 142.

We denote by $\varrho_n(\omega)$ the characteristic function of the event $B^{(n)}$:

$$\varrho_n(\omega) = \begin{cases} 1 & \text{if} \quad n \in \omega, \\ 0 & \text{if} \quad n \notin \omega. \end{cases}$$

For some $\omega = \{a_1, a_2, \ldots\} \in \Omega$, we denote by $r_n = r_n(\omega)$ the number of solutions of

$$a_x + a_y = n, \quad a_x \in \omega, \quad a_y \in \omega, \quad a_x < a_y \tag{10}$$

so that

$$|R(n) - 2r_n(\omega)| \leqq 1 \tag{11}$$

(where $R(n)$ is the number of solutions of (10) without the restriction $a_x < a_y$) and

$$r_n(\omega) = \sum_{1 \leqq j < \frac{1}{2}n} \varrho_j(\omega)\, \varrho_{n-j}(\omega).$$

Furthermore, we put

$$\delta_n(j) = \mu(\{\omega\colon j \in \omega,\ n-j \in \omega\}) = \alpha_j \alpha_{n-j} \quad \text{for} \quad j < n/2,$$

$$\lambda_n = M(r_n(\omega)) = \sum_{1 \leqq j < \frac{n}{2}} \delta_n(j)$$

(where $M(\xi)$ denotes the expectation of the random variable ξ),

$$P_n(d) = \mu(\{\omega\colon r_n(\omega) = d\}) = \tag{12}$$

$$= \sum_{1 \leqq j_1 < \ldots < j_d < n/2} \delta_n(j_1)(1-\delta_n(j_1))^{-1} \ldots \delta_n(j_d)(1-\delta_n(j_d))^{-1} \prod_{1 \leqq j < n/2} (1-\delta_n(j))$$

$$\text{for} \quad 0 \leqq d \leqq n$$

and

(13) $$f(z) = \sum_{d=0}^{n} P_n(d)\, z^d =$$

$$= \sum_{d=0}^{n} \Big(\sum_{1 \leqq j_1 < \ldots < j_d < n/2} \delta_n(j_1)(1-\delta_n(j_1))^{-1} \ldots \delta_n(j_d)(1-\delta_n(j_d))^{-1} \prod_{1 \leqq j < n/2} (1-\delta_n(j))\Big) z^d =$$

$$= \prod_{1 \leqq j < n/2} \big((1-\delta_n(j)) + \delta_n(j) z\big)$$

(for any complex number z).

We shall also need the Borel—Cantelli lemma:

LEMMA 2. *Let (X, S, μ) be a probability space and let $E_1, E_2, \ldots$ be a sequence of measurable events. If*

$$\sum_{j=1}^{+\infty} \mu(E_j) < +\infty,$$

then, with probability 1, *at most a finite number of the events E_j can occur.*

(See [4], p. 135.)

3. The proof of Theorem 2 will be based on Lemma 3 and Theorem 3 below.

LEMMA 3. *If the sequence* (8) *satisfies* (9), $n \geqq 3$, *and Δ is a real number satisfying*

(14) $$0 < \Delta < \lambda_n,$$

then we have

(15) $$\mu(\{\omega:\ |r_n(\omega) - \lambda_n| \geqq \Delta) < 2 \exp(-\Delta^2/4\lambda_n).$$

(Note that (9) implies $\lambda_n > 0$ for $n \geqq 3$.)

PROOF OF LEMMA 3. First we estimate $\mu(\{\omega:\ r_n(\omega) \geqq \lambda_n + \Delta\})$. In view of (13) and (14), for $1 < x < 2$ we have

(16) $$\mu(\{\omega:\ r_n(\omega) \geqq \lambda_n + \Delta\}) = \sum_{d \geqq \lambda_n + \Delta} P_n(d) \leqq \sum_{d \geqq \lambda_n + \Delta} P_n(d) x^{d-(\lambda_n+\Delta)} =$$

$$= x^{-(\lambda_n+\Delta)} \sum_{d \geqq \lambda_n + \Delta} P_n(d) x^d \leqq x^{-(\lambda_n+\Delta)} \sum_{d=0}^{n} P_n(d) x^d = x^{-(\lambda_n+\Delta)} f(x) =$$

$$= (1+(x-1))^{-(\lambda_n+\Delta)} \prod_{1 \leqq j < n/2} (1+(x-1)\delta_n(j)) <$$

$$< \exp\left(-(\lambda_n+\Delta)\left((x-1) - \frac{(x-1)^2}{2}\right)\right) \prod_{1 \leqq j < n/2} \exp((x-1)\delta_n(j)) =$$

$$= \exp\left(-(\lambda_n+\Delta)\left((x-1) - \frac{(x-1)^2}{2}\right) + (x-1) \sum_{1 \leqq j < n/2} \delta_n(j)\right) =$$

$$= \exp\left(-(\lambda_n+\Delta)\left((x-1) - \frac{(x-1)^2}{2}\right) + (x-1)\lambda_n\right) =$$

$$= \exp\left(-\Delta(x-1) + (\lambda_n+\Delta)\frac{(x-1)^2}{2}\right) < \exp(-\Delta(x-1) + \lambda_n (x-1)^2)$$

since we have $1+u<e^u$ for $u>0$ and

$$1+u=\exp(\log(1+u))=\exp\left(u-\frac{u^2}{2}+\frac{u^3}{3}-\dots\right)>\exp\left(u-\frac{u^2}{2}\right)\quad\text{for}\quad 0\leqq u<1.$$

Writing $x=1+\Delta/2\lambda_n$ in (16) (then $1<x<2$ holds by (14)), we obtain that

$$\text{(17)}\qquad \mu(\{\omega:\ r_n(\omega)\geqq\lambda_n+\Delta\})<\exp(-\Delta^2/2\lambda_n+\Delta^2/4\lambda_n)=\exp(-\Delta^2/4\lambda_n).$$

Similarly, for $0<x<1$ we have

$$\text{(18)}\qquad \mu(\{\omega:\ r_n(\omega)\leqq\lambda_n-\Delta\})=\sum_{d\leqq\lambda_n-\Delta}P_n(d)\leqq\sum_{d\leqq\lambda_n-\Delta}P_n(d)x^{d-(\lambda_n-\Delta)}=$$

$$=x^{-(\lambda_n-\Delta)}\sum_{d\leqq\lambda_n-\Delta}P_n(d)x^d\leqq x^{-(\lambda_n-\Delta)}\sum_{d=0}^{n}P_n(d)x^d=x^{-(\lambda_n-\Delta)}f(x)=$$

$$=(1-(1-x))^{-(\lambda_n-\Delta)}\prod_{1\leqq j<n/2}(1-(1-x)\delta_n(j))<$$

$$<\exp((1-x)(\lambda_n-\Delta))\prod_{1\leqq j<n/2}\exp\left(-(1-x)\delta_n(j)+\frac{(1-x)^2(\delta_n(j))^2}{2}\right)=$$

$$=\exp\left((1-x)(\lambda_n-\Delta)-(1-x)\sum_{1\leqq j<n/2}\delta_n(j)+\frac{(1-x)^2}{2}\sum_{1\leqq j<n/2}(\delta_n(j))^2\right)\leqq$$

$$\leqq\exp\left((1-x)(\lambda_n-\Delta)-(1-x)\sum_{1\leqq j<n/2}\delta_n(j)+\frac{(1-x)^2}{2}\sum_{1\leqq j<n/2}\delta_n(j)\right)=$$

$$=\exp\left((1-x)(\lambda_n-\Delta)-(1-x)\lambda_n+\frac{(1-x)^2}{2}\lambda_n\right)=\exp\left(-\Delta(1-x)+\frac{(1-x)^2}{2}\lambda_n\right)$$

since for $0<x<1$ we have

$$\exp(-u)<1-u=\exp(\log(1-u))=\exp\left(-u+\frac{u^2}{2}-\frac{u^3}{3}+\dots\right)<\exp\left(-u+\frac{u^2}{2}\right).$$

Writing $x=1-\Delta/\lambda_n$ in (18) (then $0<x$ holds by (14)), we obtain

$$\text{(19)}\qquad \mu(\{\omega:\ r_n(\omega)\leqq\lambda_n-\Delta\})<\exp(-\Delta^2/\lambda_n+\Delta^2/2\lambda_n)=\exp(-\Delta^2/2\lambda_n).$$

(17) and (19) yield (15).

THEOREM 3. *If the sequence* (8) *satisfies* (9), *and there exists a positive integer* n_0 *such that*

$$\text{(20)}\qquad \lambda_n=\sum_{1\leqq j<n/2}\alpha_j\alpha_{n-j}>9\log n\quad\textit{for}\quad n\geqq n_0,$$

then, with probability 1, *there exists a number* $n_1=n_1(\omega)$ *such that*

$$|R(n)-2\lambda_n|<7(\lambda_n\log n)^{1/2}\quad\text{for}\quad n>n_1.$$

PROOF. By using Lemma 3 with $\Delta=3(\lambda_n \log n)^{1/2}$ (then (14) holds by (20)), we obtain

$$\sum_{n=1}^{+\infty} \mu(\{\omega:\ |r_n(\omega)-\lambda_n| \geqq 3(\lambda_n \log n)^{1/2}\}) =$$
$$= O(1)+\sum_{n=n_0}^{+\infty} \mu(\{\omega:\ |r_n(\omega)-\lambda_n| \geqq 3(\lambda_n \log n)^{1/2}\}) <$$
$$< O(1)+2\sum_{n=n_0}^{+\infty} \exp\left(-(3(\lambda_n \log n)^{1/2})^2/4\lambda_n\right) = O(1)+2\sum_{n=n_0}^{+\infty} n^{-9/4} < +\infty.$$

Thus by the Borel—Cantelli lemma (Lemma 2), with probability 1, at most a finite number of the events

$$|r_n(\omega)-\lambda_n| \geqq 3(\lambda_n \log n)^{1/2}$$

can occur, i.e., with probability 1, there exists a number $n_2=n_2(\omega)$ such that

$$|r_n(\omega)-\lambda_n| < 3(\lambda_n \log n)^{1/2} \quad \text{for} \quad n > n_2.$$

By (11) and (20), for such a sequence ω, for large n we have

$$|R(n)-2\lambda_n| \leqq |R(n)-2r_n(\omega)|+2|r_n(\omega)-\lambda_n| < 1+6(\lambda_n \log n)^{1/2} < 7(\lambda_n \log n)^{1/2}$$

which completes the proof of Theorem 3.

4. In this section, we complete the proof of Theorem 2. We put

$$\alpha_n = \begin{cases} 1/2 & \text{for} \quad 1 \leqq n \leqq x_0, \\ g(n) & \text{for} \quad x_0 < n < +\infty. \end{cases}$$

Defining the sequence (8) in this way, (9) holds trivially. Furthermore, in view of (iii) in Theorem 2, we have

$$\lambda_n = \sum_{1\leqq j<n/2} \alpha_j \alpha_{n-j} = \sum_{x_0<j\leqq n/2} g(j)g(n-j)+O(1). \tag{21}$$

By (i) in Theorem 2, we may use the Euler—Maclaurin summation formula in order to estimate the last sum. In view of (i), (ii) and (iii), we obtain

(22)

$$\sum_{x_0<j\leqq n/2} g(j)g(n-j) = \int_{x_0}^{n/2} g(x)g(n-x)dx - \left[g(x)g(n-x)\left(x-[x]-\frac{1}{2}\right)\right]_{x_0}^{n/2} +$$
$$+\int_{x_0}^{n/2} (g'(x)g(n-x)-g(x)g'(n-x))\left(x-[x]-\frac{1}{2}\right)dx =$$
$$= \left(\int_1^{n/2} g(x)g(n-x)dx+O(1)\right)+O\left((g(n/2))^2+g(x_0)g(n-x_0)\right)+$$
$$+O\left(\int_{x_0}^{n/2} (|g'(x)|+|g'(n-x)|)dx\right) =$$
$$= \int_1^{n/2} g(x)g(n-x)dx+O(1)+O\left(\int_{x_0}^{n/2} (-g'(x)-g'(n-x))dx\right) =$$
$$= \int_1^{n/2} g(x)g(n-x)dx+O(1)+O\left([-g(x)+g(n-x)]_{x_0}^{n/2}\right) = \int_1^{n/2} g(x)g(n-x)\,dx+O(1).$$

(21) and (22) yield

$$\lambda_n = \int_1^{n/2} g(x)g(n-x)dx + O(1).$$

Thus by (7) and (iv) in Theorem 2,

$$(23)\quad \left|F(n)-2\lambda_n\right| \leqq |F(n)-2\int_1^{n/2} g(x)g(n-x)\,dx| + 2\left|\int_1^{n/2} g(x)\,g(n-x)dx - \lambda_n\right| <$$
$$< \big(F(n)\log n\big)^{1/2} + O(1) < 2\big(F(n)-\log n\big)^{1/2}$$

for large n, hence in view of (7),

$$\lambda_n > \frac{1}{2}F(n) - \big(F(n)\log n\big)^{1/2} > \frac{1}{2}F(n) - \left(F(n)\cdot\frac{F(n)}{36}\right)^{1/2} = \frac{1}{3}F(n) > 12\log n$$

so that also (20) holds.

Thus all the conditions in Theorem 3 hold. By using Theorem 3, we obtain that, with probability 1, for large n we have

$$(24)\qquad |R(n)-2\lambda_n| < 7(\lambda_n \log n)^{1/2}.$$

In view of (7), (23) and (24) yield for large n

$$|R(n)-F(n)| \leqq |R(n)-2\lambda_n| + |2\lambda_n - F(n)| < 7(\lambda_n \log n)^{1/2} + |2\lambda_n - F(n)| \leqq$$
$$\leqq 7\left(\left(\frac{1}{2}F(n)+\frac{1}{2}|2\lambda_n - F(n)|\right)\log n\right)^{1/2} + |2\lambda_n - F(n)| <$$
$$< 7\left(\left(\frac{1}{2}F(n)+\big(F(n)\log n\big)^{1/2}\right)\log n\right)^{1/2} + 2\big(F(n)\log n\big)^{1/2} <$$
$$< 7\left(\left(\frac{1}{2}F(n)+\left(F(n)\cdot\frac{F(n)}{36}\right)^{1/2}\right)\log n\right)^{1/2} + 2\big(F(n)\log n\big)^{1/2} =$$
$$= 7\left(\frac{2}{3}F(n)\log n\right)^{1/2} + 2\big(F(n)\log n\big)^{1/2} < 8\big(F(n)\log n\big)^{1/2}$$

which completes the proof of Theorem 3.

5. So far we have estimated the probabilities $P_n(d)$ for d "far" from the expectation $\lambda_n = M\big(r_n(\omega)\big)$. In [2], Erdős and Rényi gave lower and upper bounds for $P_n(d)$ for *all* d. These estimates give the right order of magnitude of $P_n(d)$ for d "near" λ_n, provided $\alpha_j = O(j^{-1/4})$. Furthermore, they determined the limit distribution of $r_n(\omega)$. Sharpening and generalizing these estimates, we are going to complete this paper by giving an asymptotics for $P_n(d)$ for d "near" λ_n.

THEOREM 4. *If the sequence* (8) *satisfies* (9),

$$(25)\qquad \lim_{n\to+\infty} \alpha_n = 0$$

and

(26) $$\lambda_n = \sum_{1 \leqq j < n/2} \alpha_j \alpha_{n-j} > 3 \quad \text{for} \quad n \geqq n_0,$$

and we put

$$\lambda_n' = \sum_{1 \leqq j < n/2} \alpha_j \alpha_{n-j}(1-\alpha_j\alpha_{n-j}) = \sum_{1 \leqq j < n/2} \delta_n(j)\big(1-\delta_n(j)\big),$$

then for $n > n_1$ *and all* d *we have*

$$\left| P_n(d) - \frac{1}{(2\pi\lambda_n')^{1/2}} e^{-(\lambda_n-d)^2/2\lambda_n'} \right| < 13 \frac{(\log \lambda_n)^2}{\lambda_n}$$

where $P_n(d)$ *is defined by* (12).

(Thus the limit distribution of the random variable $\frac{r_n(\omega)-\lambda_n}{(\lambda_n')^{1/2}}$ is the normal distribution.)

Proof. Throughout the proof, θ will denote a complex number with absolute value $\leqq 1$. (In other words, $u = \theta v$ means that $|u| \leqq |v|$.)

We denote the characteristic function of the random variable $r_n(\omega)$ by $\varphi(t)$, so that in view of (13)

$$\varphi(t) = M(e^{ir_n(\omega)t}) = f(e^{it}) = \sum_{d=0}^{n} P_n(d) e^{idt} = \sum_{1 \leqq j < n/2} \big((1-\delta_n(j)) + \delta_n(j) e^{it}\big).$$

Furthermore, we put

$$\eta = 2\left(\frac{\log \lambda_n}{\lambda_n}\right)^{1/2}.$$

Then we have

(27) $$P_n(d) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varphi(t) e^{-idt}\, dt =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(1/2)\lambda_n' t^2} e^{i(\lambda_n-d)t}\, dt - \frac{1}{2\pi} \int_{\eta \leqq |t|} e^{-(1/2)\lambda_n' t^2} e^{i(\lambda_n-d)t}\, dt +$$

$$+ \frac{1}{2\pi} \int_{|t| \leqq \eta} (e^{-i\lambda_n t}\varphi(t) - e^{-(1/2)\lambda_n' t^2}) e^{i(\lambda_n-d)t}\, dt +$$

$$+ \frac{1}{2\pi} \int_{\eta \leqq |t| \leqq \pi} \varphi(t) e^{-idt}\, dt = J - J_1 + J_2 + J_3.$$

First we estimate J. Substituting $t = (\lambda_n')^{-1/2} x$, we obtain

(28) $$J = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(1/2)\lambda_n' t^2} e^{i(\lambda_n-d)t}\, dt =$$

$$= \frac{1}{2\pi(\lambda_n')^{1/2}} \int_{-\infty}^{+\infty} e^{-(1/2)x^2} e^{i(\lambda_n-d)(\lambda_n')^{-1/2}x}\, dx = \frac{1}{(2\pi\lambda_n')^{1/2}} e^{-(\lambda_n-d)^2/2\lambda_n'}$$

since it is well-known that

$$\int_{-\infty}^{+\infty} e^{iux-x^2/2}dx = (2\pi)^{1/2}e^{-u^2/2}$$

(see, e.g., [5], p. 261).

In order to estimate J_2 and J_3, we need an estimate for $\varphi(t)$. For $|z|<1/2$, we have

$$\left|e^z-\left(1+z+\frac{z^2}{2}\right)\right| = \left|\sum_{k=3}^{+\infty}\frac{z^k}{k!}\right| \leqq \sum_{k=3}^{+\infty}\frac{|z|^k}{3} = \frac{|z|^3}{6}\frac{1}{1-|z|} < \frac{|z|^2}{3}$$

and

$$1-z = \exp(\log(1-z)) = \exp\left(-z+\frac{z^2}{2}-\frac{z^3}{3}+\dots\right) =$$

$$= \exp\left(-z+\theta\left(\frac{|z|^2}{2}+\frac{|z|^3}{3}+\dots\right)\right) = \exp\left(-z+\theta\left(\frac{|z|^2}{2}+\frac{|z|^3}{2}+\dots\right)\right) =$$

$$= \exp\left(-z+\theta\frac{|z|^2}{2}\frac{1}{1-|z|}\right) = \exp(-z+\theta|z|^2).$$

Thus in view of (25), for large n, $1\leqq j<n/2$ and $|t|\leqq 1/2$ we have

$$e^{-i\delta_n(j)t}((1-\delta_n(j))+\delta_n(j)e^{it}) = e^{-i\delta_n(j)}(1-\delta_n(j)(1-e^{it})) =$$

$$= \left(1-i\delta_n(j)t-\frac{1}{2}(\delta_n(j))^2t^2+\frac{\theta}{3}(\delta_n(j))^3t^3\right)\left(1-\delta_n(j)\left(-it+\frac{t^2}{2}+\frac{\theta}{3}t^3\right)\right) =$$

$$= 1-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3 =$$

$$= \exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3+\theta\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3\right)^2\right) =$$

$$= \exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\frac{\theta}{2}\delta_n(j)t^3+\frac{2\theta}{3}(\delta_n(j))^2t^4\right) =$$

$$= \exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\theta\delta_n(j)t^3\right)$$

hence

$$(29) \qquad e^{-i\lambda_n t}\varphi(t) = \prod_{1\leqq j<n/2} e^{-i\delta_n(j)t}((1-\delta_n(j))+\delta_n(j)e^{it}) =$$

$$= \prod_{1\leqq j<n/2}\exp\left(-\frac{1}{2}(\delta_n(j)-(\delta_n(j))^2)t^2+\theta\delta_n(j)t^3\right) = e^{-(1/2)\lambda_n' t^2+\theta\lambda_n t^3}$$

(for large n and $|t|\leqq 1/2$).

Furthermore, in view of (25), for large n and $|t|\leqq\pi$ we have

$$|\varphi(t)| = \prod_{1\leqq j<n/2} |1-\delta_n(j)+\delta_n(j)e^{it}| = \tag{30}$$

$$= \prod_{1\leqq j<n/2} ((1-\delta_n(j)+\delta_n(j)e^{it})(1-\delta_n(j)+\delta_n(j)e^{-it}))^{1/2} =$$

$$= \prod_{1\leqq j<n/2} ((1-\delta_n(j))^2+(\delta_n(j))^2+2\delta_n(j)(1-\delta_n(j))\cos t)^{1/2} =$$

$$= \prod_{1\leqq j<n/2} (1+2\delta_n(j)(1-\delta_n(j))(\cos t-1))^{1/2} =$$

$$= \prod_{1\leqq j<n/2} (1-4\delta_n(j)(1-\delta_n(j))(\sin t/2)^2)^{1/2} \leqq$$

$$\leqq \prod_{1\leqq j<n/2} \left(1-3\delta_n(j)\left(\frac{2}{\pi}\cdot\frac{t}{2}\right)^2\right)^{1/2} = \prod_{1\leqq j<n/2} \left(1-\frac{3}{\pi^2}\delta_n(j)t^2\right)^{1/2} \leqq$$

$$\leqq \prod_{1\leqq j<n/2} \left(1-\frac{1}{4}\delta_n(j)t^2\right)^{1/2} < \prod_{1\leqq j<n/2} \left(1-\frac{1}{8}\delta_n(j)t^2\right) <$$

$$< \prod_{1\leqq j<n/2} e^{-(1/8)\delta_n(j)t^2} = e^{-(1/8)\lambda_n t^2}$$

(for large n and $|t|\leqq\pi$), since

$$|\sin x| \geqq \frac{2}{\pi}|x| \quad \text{for} \quad |x| \leqq \pi/2,$$

$$(1-u)^{1/2} < 1-\frac{u}{2} \quad \text{for} \quad 0\leqq u<1$$

and

$$(0<)\ 1-x<e^{-x} \quad \text{for} \quad 0\leqq x<1.$$

By (25), (29) and (30). for large n we have

$$|J_1|+|J_3| \leqq \frac{1}{2\pi}\int_{\eta\leqq|t|} e^{-(1/2)\lambda_n' t^2}\,dt+\frac{1}{2\pi}\int_{\eta\leqq|t|\leqq\pi} |\varphi(t)|\,dt \leqq \tag{31}$$

$$\leqq \frac{1}{2\pi}\Big(\int_{\eta\leqq|t|} e^{-(1/2)\lambda_n' t^2}dt+\int_{\eta\leqq|t|\leqq\pi} e^{-(1/8)\lambda_n t^2}dt\Big) \leqq$$

$$\leqq \frac{1}{2\pi}\Big(\int_{\eta\leqq|t|} e^{-(1/8)\lambda_n t^2}dt+\int_{\eta\leqq|t|} e^{-(1/8)\lambda_n t^2}dt\Big) = \frac{1}{\pi}\int_{\eta\leqq|t|} e^{-(1/8)\lambda_n t^2}dt =$$

$$= \frac{2}{\pi}\int_{\eta}^{+\infty} e^{-(1/8)\lambda_n t^2}dt \leqq \frac{2}{\pi}\int_{\eta}^{+\infty} \frac{t}{\eta}e^{-(1/8)\lambda_n t^2}dt =$$

$$= -\frac{8}{\pi\eta\lambda_n}[e^{-(1/8)\lambda_n t^2}]_{\eta}^{+\infty} = \frac{8}{\pi\eta\lambda_n}e^{-(1/8)\lambda_n\eta^2} =$$

$$= \frac{4}{\pi(\lambda_n\log\lambda_n)^{1/2}}e^{-(1/2)\log\lambda_n} < \frac{2}{\lambda_n(\log\lambda_n)^{1/2}}$$

and

(32)
$$|J_2| \leqq \frac{1}{2\pi} \int_{|t| \leqq \eta} |e^{-i\lambda_n t}\varphi(t) - e^{-(1/2)\lambda_n' t^2}|\, dt = \frac{1}{2\pi} \int_{|t| \leqq \eta} e^{-(1/2)\lambda_n' t^2} |e^{\theta(t)\lambda_n t^3} - 1|\, dt \leqq$$
$$\leqq \frac{1}{2\pi} \int_{|t| \leqq \eta} 2\lambda_n |t|^3\, dt \leqq \frac{1}{\pi}\lambda_n \int_{|t| \leqq \eta} \eta^3\, dt = \frac{2}{\pi}\lambda_n \eta^4 < 11\, \frac{(\log \lambda_n)^2}{\lambda_n}$$

since
$$|e^z - 1| = \left| z + \frac{z^2}{2!} + \frac{z_3}{3!} + \dots \right| \leqq |z| + |z|^2 + |z|^3 + \dots = \frac{|z|}{1-|z|} \leqq 2|z|.$$

In view of (26), (27), (28), (31) and (32) yield for large n that
$$|P_n(d) - J| = \left| P_n(d) - \frac{1}{(2\pi\lambda_n')^{1/2}} e^{-(\lambda_n - d)^2/2\lambda_n'} \right| \leqq$$
$$\leqq |J_1| + |J_2| + |J_3| < \frac{2}{\lambda_n (\log n)^{1/2}} + 11\, \frac{(\log \lambda_n)^2}{\lambda_n} < 13\, \frac{(\log \lambda_n)^2}{\lambda_n}$$

which completes the proof of Theorem 4.

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