MULTIPLICATIVE FUNCTIONS AND SMALL DIVISORS

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1. Introduction[3]

Let S be a set of positive integers and g be a multiplicative function. Consider the problem of estimating the sum

$$S(x,g) = \sum_{\substack{n \leq x \\ n \in S}} g(n). \tag{1.1}$$

A natural way to start is to write

$$g(n) = \sum_{d|n} h(d) \tag{1.2}$$

and reverse the order of summation. This in turn leads to the estimation of the contribution arising from the large divisors d of n, where n S, which often presents difficulties. In this paper we shall characterize in various ways the following idea:

"*Large divisors of a square-free integer have more prime divisors than the small ones.*" (1.3)

When the multiplicative function h is small in size, (1.3) will be useful in several situations to show that the principal contribution is due to the small divisors. The terms `large´ and `small´ will be made precise in the sequel.

An application to Probabilistic Number Theory is discussed in

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[3]As this paper evolved we had several useful discussions with Amit Ghosh, Roger Heath-Brown and Michael Vose.

Sec.4; indeed, it was this application which motivated the present paper (see [1], [3]). Our discussion in the first two sections is quite general - in Sec.2 the principal result is derived for sets rather than for divisors only and in Sec.3 the main inequality is for submultiplicative functions. This is done in the hope that our elementary methods may have other applications as well, perhaps even outside of Number Theory.

2. A mapping for sets.

If n is not square it is trivial to note that half its divisors are less than $\sqrt{n}$. If n is square-free there is also an interesting one-to-one correspondence, namely: there is a bijective mapping m between the divisors d of n which are less than $\sqrt{n}$ and the divisors d′ of n which are greater than $\sqrt{n}$ such that

$$m(d) = d' \equiv 0 \pmod{d} \qquad (2.1)$$

(of course the mapping m depends on n). In fact, this mapping is a special case of a rather general one-to-one correspondence that can be set up between subsets of a finite set, as we shall presently see.

Let $\mathbf{S}$ be a finite set and λ a finite measure on the set of all subsets of $\mathbf{S}$. For each $t > 0$ define

$$A(t,\mathbf{S}) = \{\mathbf{E} \subseteq \mathbf{S} : \lambda(\mathbf{E}) < t\}.$$

We then have

Theorem 1. *For each $t > 0$ there is a permutation*

$$\pi_{t,\mathbf{S}} : A(t,\mathbf{S}) \to A(t,\mathbf{S})$$

such that for all $\mathbf{E} \in A(t,\mathbf{S})$ we have $\pi_{t,\mathbf{S}}(\mathbf{E}) \cap \mathbf{E} = \emptyset$.

Remark. There are trivial cases here. If $\lambda(\mathbf{S}) < t$ then $A(t,\mathbf{S})$ is the power set of $\mathbf{S}$ and so the permutation $\mathbf{E} \to \mathbf{S} - \mathbf{E}$ has the desired property. If $t = 0$ then $A(0,\mathbf{S})$ is the power set of $\mathbf{S}^{(0)}$ where

$S^{(0)} = \{ s \in S : \lambda(s) = 0\}$. Here $E \to S^{(0)} - E$ is an appropriate permutation. So in the proof that follows we assume that $0 < t < \lambda(S)$.

Proof. If S has cardinality $|S| = 1$ the result is trivial. We proceed by induction of $|S|$.

Let $|S| = N \geq 2$ and assume the result is true for sets with $N - 1$ elements. Pick x in S with $\lambda(\{x\}) \leq t$. (If such an x does not exist the result is trivially true because $A(t,S) = \emptyset$.) Let $T = S - \{x\}$ and note that $|T| = N - 1$. By our inductive hypothesis there is for each $\tau \geq 0$, a permutation $\pi_{\tau,T}$ of $A(\tau,T)$ such that

$$\pi_{\tau,T}(E) \cap E = \emptyset \text{ for all } E \subseteq A(\tau,T).$$

We partition $A(t,S)$ into three disjoint subsets as follows:

$$A_1(t,S) = \{E \subseteq A(t,S) \mid x \in E\},$$

$$A_2(t,S) = \{E \subseteq A(t,S) \mid x \notin E,\ t - \lambda(\{x\}) < \lambda(E) \leq t\},$$

$$A_3(t,S) = \{E \subseteq A(t,S) \mid x \notin E,\ \lambda(E) \leq t - \lambda(\{x\})\}.$$

Next, define

$$\phi : A_1(t,S) \cup A_2(t,S) \to A(t,T)$$

by $\phi(E) = E - \{x\}$ and

$$\psi : A(t - \lambda(\{x\}),T) \to A_1(t,S)$$

by $\psi(E) = E \cup \{x\}$. Clearly both ϕ and ψ are bijective. Also

$$A_2(t,S) \cup A_3(t,S) = A(t,T)$$

and

$$A_3(t,S) = A(t - \lambda(\{x\}),\ T).$$

We define $\pi_{t,S}$ as follows

$$\pi_{t,S}(E) = \begin{cases} \pi_{t,T}(\phi(E)) & \text{if } E \in A_1(t,S) \cup A_2(t,S) \\ \phi(\pi_{t-\lambda(\{x\}),T}(E)) & \text{if } E \in A_3(t,S). \end{cases}$$

It is easy to check that $\pi_{t,S}$ has the desired properties and this proves Theorem 1.

Corollary. *Let* S, λ *be as above. Define*

$$B(t,S) = \{E \subseteq S : \lambda(S) - t \leq \lambda(E)\};$$

then there is a bijection $\sigma_{t,S} : A(t,S) \to B(t,S)$ *such that* $E \subseteq \sigma_{t,S}(E)$ *for all* $E \in A(t,S)$.

Proof. Define $\sigma_{t,S}(E) = S - \pi_{t,S}(E)$ and use Theorem 1.

Let $n = p_1 \cdots p_r$ be square-free and $S = \{p_1, p_2, \ldots, p_r\}$ with $\lambda(p_i) = \log p_i$, $i = 1,2,\ldots,r$. We apply the Corollary with this choice of S and λ (and with t replaced by $\log t$) to obtain the following result, which, in view of its number theoretic form, is given the status of a theorem.

Theorem 2. *Let* n *be square-free and* $t \geq 1$. *Then there is a one-to-one mapping* m_t *between the divisors* d *of* n *which are less than or equal to* t *and those divisors* d' *of* n *which are greater than or equal to* n/t, *such that*

$$m_t(d) = d' \equiv 0 \pmod{d}.$$

Remarks.

1.) In Theorem 2 the parameter t could be greater than $\sqrt{n}$, but only $t \leq \sqrt{n}$ is of interest here. If $t > \sqrt{n}$ then $\tau = n/t < \sqrt{n}$. In this case m_τ produces a correspondence between $d \leq \tau$ and $d' \geq t$. The divisors between τ and t can be mapped onto themselves and m_t for $t > \sqrt{n}$ can be easily constructed from m_τ, where $\tau < \sqrt{n}$.

2.) The case $t = \sqrt{n}$ is of special interest because it shows that for a multiplicative function h satisfying $0 \le h \le 1$ we have

$$\sum_{d|n} h(d) \le 2 \sum_{\substack{d|n \\ d \le \sqrt{n}}} h(d), \text{ for all square-free } n. \tag{2.2}$$

Note that (2.2) is an immediate consequence of (2.1) (which is Theorem 2 with $t = \sqrt{n}$) because $h(d') \le h(d)$. Inequality (2.2) can be proved directly without use of (2.1) as was pointed out by Heath-Brown. For this direct proof and applications see [3], [1].

3.) In a private correspondence to one of us (K.A.) R.R. Hall reported that Woodall had arrived at the mapping (2.1) a few years ago. Never-the-less, applications of such mappings or inequalities to Probabilistic Number Theory in [1], [3], appear to be new.

4.) When $h \ge 1$, clearly (2.2) is false. In fact, in this case (2.2) does not even hold if 2 is replaced by an arbitrarily large constant. Note that the constant 2 is best possible in (2.2) by taking $h \equiv 1$.

3. A useful inequality.

In view of (2.2) we may ask as to what sort of conditions one should impose upon h so that for all square-free n,

$$\sum_{d|n} h(d) \ll_k \sum_{\substack{d|n \\ d \le n^{1/k}}} h(d), \tag{3.1}$$

where $k \ge 2$. Because of (1.3) we may expect (3.1) to hold provided $h(p)$ is quite small.

To get an idea concerning the size of such h we consider the special multiplicative function with $h(p) = c > 0$. Let r be a large integer and $p_1, p_2, \ldots, p_r$ primes such that $p_1 \sim p_2 \sim p_3 \sim \ldots \sim p_r$. Let $n = p_1 p_2 \ldots p_r$. In this situation a divisor d of n satisfies $d \le n^{1/k}$ provided d has (asymptotically) $\le r/k$ prime factors. Thus,

$$\Big\{ \sum_{d|n} h(d) \Big\}\Big\{ \sum_{\substack{d|n \\ d \le n^{1/k}}} h(d) \Big\}^{-1} \sim (1+c)^r \Big\{ \sum_{\ell=0}^{r/k} \binom{r}{\ell} c^\ell \Big\}^{-1} . \quad (3.2)$$

The maximum value of $\binom{r}{\ell}c^\ell$ occurs when $\ell \sim rc/(c+1)$, as $r \to \infty$. So the left hand side of (3.2) is unbounded if $c/(c+1) > 1/k$, i.e., if $c > 1/(k-1)$. On the other hand if $c < 1/(k-1)$ then the expressions in (3.2) are ~ 1 as $r \to \infty$. This example led one of us (K.A.) to make the following conjecture, part (i) of which appeared as problem 407 in the *West Coast Number Theory Conference, Asilomar (1983)*:

Conjecture.

(i) *For each* $k \ge 2$, *there exists a constant* c_k *such that* (3.1) *holds for all multiplicative functions* h *satisfying* $0 \le h(p) \le c_k$, *for all* p.

(ii) *In part* (i) $c_k = 1/(k-1)$ *is admissible.*

To this end we now prove an inequality for certain submultiplicative functions h, namely, those h for which $h(mn) \le h(m)h(n)$, if $(m,n) = 1$.

Theorem 3. *Let* $h \ge 0$ *be submultiplicative and satisfy* $0 \le h(p) \le c < 1/(k-1)$ *for all primes* p. *Then for all square-free* n *we have*

$$\sum_{d|n} h(d) \le \Big\{ 1 - \frac{kc}{1+c} \Big\}^{-1} \sum_{\substack{d|n \\ d \le n^{1/k}}} h(d) .$$

Proof. We begin with the familiar decompositon

$$\sum_{d|n} h(d) = \sum_{d|\,n/p} h(d) + \sum_{d|\,n/p} h(pd),$$

where p is any prime divisor of n. Submultiplicativity yields

$$h(p) \sum_{d|n} h(d) = h(p) \sum_{d|\,n/p} h(d) + h(p) \sum_{d|\,n/p} h(pd) \quad (3.3)$$

$$\ge \sum_{d|\,n/p} h(np) + h(p) \sum_{d|\,n/p} h(pd)$$

$$= (1 + h(p)) \sum_{d | n/p} h(d) .$$

Next, observe that

$$\sum_{\substack{d|n \\ d\le n^{1/k}}} h(d) \ge \sum_{d|n} h(d) \frac{\log(n^{1/k}/d)}{\log(n^{1/k})} = \sum_{d|k} h(d) - \frac{k}{\log n} \sum_{d|n} h(d)\log d. \tag{3.4}$$

In addition

$$\sum_{d|n} h(d) \log d = \sum_{d|n} h(d) \sum_{p|d} \log p = \sum_{p|n} \log p \sum_{d | n/p} h(pd)$$
$$\le \{ \sum_{d|n} h(d)\}\{ \sum_{p|n} \frac{(\log p) h(p)}{1 + h(p)} \} \tag{3.5}$$

because of (3.3). Since $0 \le h(p) \le c$, we have $h(p)/(1 + h(p))$ $\le c/(1 + c)$. By combining (3.4) and (3.5) we obtain Theorem 3.

Remarks.

1.) Theorem 3 proves Conjecture (i) for any $c_k < 1/(k-1)$. The case $c_k = 1/(k-1)$ (part (ii)) is still open when $k > 2$ (for $k = 2$ this is (2.2)). The analysis underlying (3.2) shows that $c_k > 1/(k-1)$ is not possible.

2.) It would be of interest to see if the constant $(1 - \frac{kc}{1+c})^{-1}$ can be improved. An attempt to deal with the case $c_k = 1/(k-1)$ may throw some light on this question.

3.) R. Balasubramaniam and S. Srinivasan (personal communication to one of us - K.A.) have obtained slightly weaker versions of Theorem 3 in response to our conference query in the course of proving Conjecture (i) for $c_k < 1/(k-1)$.

4.) If h is submultiplicative, then so is $h_T(n)$ which is equal to $h(n)$ when $n \le T$ and is zero for $n > T$. The proof of Theorem 3 shows

$$\sum_{\substack{d|n \\ d \le T}} h(d) < \{ \sum_{\substack{d|n \\ d\le t}} h(d)\}\{1 - \frac{1}{\log t} \sum_{\substack{p|n \\ p\le T}} \frac{h(p) \log p}{1 + h(p)} \}^{-1}$$

holds uniformly for all square-free $0 < t < T$ and submultiplicative h satisfying $h \geq 0$ and $0 \leq h(p) \leq (\log t)/(\log n/t)$.

5.) Let h be super-multiplicative, that is, $h(mn) \geq h(m)h(n)$, for $(m,n) = 1$. Suppose $h(p) \geq c > 1/(k-1)$ for all primes p. Then the proof of Theorem 3 can be modified to yield the dual inequality

$$\sum_{d|n} h(d) \geq \frac{(1+c)(k-1)}{k} \sum_{\substack{d|n \\ d \leq n^{1/k}}} h(d)$$

for all square-free n. Here also the situation regarding $c = 1/(k-1)$ is open.

4. An application.

Let S be an infinite set of positive integers. Define

$$S_d(x) = \sum_{\substack{s \leq x,\ s \in S \\ s \equiv 0 \pmod d}} 1 ,$$

and set $X = S_1(x)$. In addition, let

$$S_d(x) = \frac{X\omega(d)}{d} + R_d(x) ,$$

where ω is multiplicative. First we assume that R_d satisfies the following condition:

(C-1) There exists $\delta > 0$ such that uniformly in x

$|R_d(x)| \ll \frac{X\omega(d)}{d}$ (equivalently $S_d(x) \ll \frac{X\omega(d)}{d}$) for $1 \leq d \leq x^{\delta}$.

We also require R_d to satisfy at least one of the following two conditions:

(C-2) $|R_d(x)| \ll \omega(d)$, or

(C-3) There exist $\beta > 0$ such that to each $U > 0$ there is $V > 0$ satisfying

$$\sum_{d \leq X^{\beta}/(\log X)^{V}} |R_d(x)| \ll_U X/(\log^U X) .$$

Furthermore, we also require that there exists $c > 0$ such that

$$|R_d(x)| \ll (\frac{X \log X}{d} + 1)\, c^{\nu(d)}, \quad 1 \leq d \leq x,$$

where $\nu(n) = \sum_{p|n} 1$.

Examples of sets S satisfying these conditions include

(E-1) $S = \{Q(n) \mid n = 1,2,3,\ldots\}$, where $Q(x)$ is a polynomial with positive integer coefficients. Here $\omega(d) = \rho(d)$, the number of solutions of $Q(x) \equiv 0 \pmod d$ and $|R_d| \leq \rho(d)$, so (C-2) holds. We may take $\delta = 1/(\deg Q)$ in (C-1).

(E-2) $S = \{p + a \mid p = \text{prime}\}$, where a is a fixed positive integer. Here $\omega(d) = d/\phi(d)$ where ϕ is Euler's function. By the Brun-Titchmarsh inequality (see Halberstam-Richert [6], p.107) we can take any $\delta \in (0,1)$ in (C-1). By Bombieri's theorem (see [6], p. 111), we see that (C-3) holds with $\beta = 1/2$.

Let f be a (complex valued) strongly additive function, namely, one that satisfies

$$f(n) = \sum_{p|n} f(p).$$

The quantities

$$A(x) = \sum_{p \leq x} \frac{f(p)\omega(p)}{p} \quad \text{and} \quad B(x) = \sum_{p \leq x} \frac{|f^2(p)|\omega(p)}{p}$$

act like the 'mean' and 'variance' of $f(n)$, for $n \in S$, $n \leq x$. Our problem is to obtain a bound for

$$\sum_{\substack{n \leq x \\ n \in S}} |f(n) - A(x)|^{\ell}, \quad \ell = 1,2,3,\ldots$$

in terms of $B(x)$. In the special case where S is the set of all positive integers, Elliott [4] has solved this problem elegantly.

Recently one of us (K.A.) has improved Elliott's method in order to make it applicable to subsets. In [2] sets **S** with $\delta = 1$ in (C-1) are treated whereas in [3] the situation concerning **S** in (E-2) is investigated. It is this improved method which we shall employ here; we sketch only the main ideas since details may be found in [2], [3].

We start by introducing a simplification: We may assume that $f \geqslant 0$. This is because the inequality

$$|a + b|^{\ell} \ll_{\ell} |a|^{\ell} + |b|^{\ell} \tag{4.1}$$

is valid for all complex numbers a and b. So a complex function could be decomposed into its real and imaginary parts. If f is real valued we can write $f = f^{+} - f^{-}$, where f^{+}, f^{-} are non-negative strongly additive functions generated by

$$f^{+}(p) = \max(0, f(p)), \quad f^{-}(p) = -\min(0, f(p)).$$

For convenience we introduce the distribution function

$$F_x(\nu) = \frac{1}{X} \sum_{\substack{n \leq x,\ n \in \mathbf{S} \\ f(n) - A(x) \leq \nu\sqrt{B(x)}}} 1 .$$

We note that for even ℓ

$$\frac{1}{XB(x)^{\ell/2}} \sum_{\substack{n \leq x \\ n \in \mathbf{S}}} (f(n) - A(x))^{\ell} = \int_{-\infty}^{\infty} \nu^{\ell} dF_x(\nu) . \tag{4.2}$$

Our aim is to show that the moments of F_x are bounded (uniformly in x).

To accomplish this we consider the bilateral Laplace transform

$$T_u(x) = \int_{-\infty}^{\infty} e^{uv} dF_x(v) .$$

If there is $R > 0$ for which $T_u(x) \ll 1$ when $|u| \leq R$, then it follows that the expression in (4.2) is bounded. Note that

$$T_u(x) = \frac{1}{X} \sum_{\substack{n \leq x \\ n \in S}} e^{u(f(n) - A(x)/\sqrt{B(x)})} = \frac{e^{-uA(x)/\sqrt{B(x)}}}{X} \sum_{\substack{n \leq x \\ n \in S}} g(n),$$

where

$$g(n) = e^{uf(n)/\sqrt{B(x)}} . \tag{4.3}$$

Of course g is strongly multiplicative (that is $g(n) = \prod_{p|n} g(p)$), because f is strongly additive. Our goal therefore is to bound $S(x,g)$ (see (1.1)) suitably. We have two cases.

Case 1: $u \leq 0 \Rightarrow 0 < g \leq 1$.

In a recent paper [2] it was shown by using a sieve method, that in Case 1, for the sets S satisfying either (C-2) or (C-3), we have

$$S(x,g) \ll x \prod_{p\leq x} \left(1 + \frac{(g(p) - 1)\omega(p)}{p}\right) . \tag{4.4}$$

Case 2: $u > 0 \Rightarrow g \geq 1$.

Here we let $\delta = 1/k$ (in C-1) and assume that f satisfies

$$\{ \max_{p\leq x} f(p) \} /\sqrt{B(x)} \ll 1. \tag{4.5}$$

Then we can choose $R > 0$ (sufficiently small) such that

$$1 \leq g(p) \leq 1 + \frac{1}{2(k-1)} .$$

With h as in (1.2) we note that $0 \leq h(p) = g(p) - 1 \leq \frac{1}{2(k-1)}$. Also $h(p^e) = 0$ for all p, $e \geq 2$, because g is strongly multiplicative. So by Theorem 3

$$\begin{aligned} S(x,g) = \sum_{\substack{n\leq x \\ n\in S}} \sum_{d|n} h(d) &\ll \sum_{\substack{n\leq x \\ n\in S}} \sum_{\substack{d|n_\delta \\ d\leq n^\delta}} h(d) \\ &\leq \sum_{d\leq x^\delta} h(d) S_d(x) . \end{aligned}$$

By (C-1) we obtain

$$S(x,g) \ll x \sum_{d\leq x^\delta} \frac{h(d)\omega(d)}{d} \leq x \prod_{p\leq x} \left(1 + \frac{h(p)\omega(p)}{p}\right) . \tag{4.6}$$

Inequalities (4.4) and (4.6) combine with (4.3) and (4.5) to yield

$T_u(x) \ll 1$ for $|u| \leq R$. For details relating to such calculations see [2], Sec.7 . Therefore by means of this method we obtain the following extension of a result of Elliott [4],

Theorem 4. *Let* f *be as above and* $|f|$ *satisfy* (4.5). *Then*

$$\sum_{\substack{n \leq x \\ n \in S}} |f(n) - A(x)|^{\ell} \ll_{\ell} XB(x)^{\ell/2}\ , \ \textit{for all } \ell > 0\ .$$

Remarks.

1.) Although our discussion was for even ℓ, Theorem 4 is stated for all $\ell > 0$. This is because one can pass from even ℓ to all positive real numbers by a suitable application of the Hölder-Minkowski inequality.

2.) If f satisfies certain additional conditions then one can use the above method more carefully to obtain asymptotic estimates for the moments. In these cases the weak limit of $F_x(v)$ would exist. Such asymptotic estimates are obtained in [2] for S with $\delta = 1$, and in [3] for S in (E-2). For these sets the full strength of Theorem 3 is not required. The inequality (2.2) (which follows from Theorem 2) suffices.

3.) There are certain open problems concerning the behavior of additive functions in polynomial sequences (see Elliott [5], Vol. 2, p. 335). Part of the difficulty in such questions is because we do not fully understand the moments of additive functions in these sequences. Theorem 4 is derived in the hope that it might shed some light on these questions.

4.) We restrict our attention to strongly additive functions for the sake of simplicity. From here the transition to general additive functions is not difficult. This procedure for the case $\delta = 1$ is illustrated in [2], Sec.10.

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