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ON THE RESIDUES OF PRODUCTS OF PRIME NUMBERS

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1.

Throughout this paper, we use the following notations:

$c_1, c_2, \dots$ denote positive absolute constants. $\Lambda(n)$ is Mangoldt's symbol:

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^\alpha , \\ 0 \text{ otherwise.} \end{cases}$$

q is a large but fixed prime number, a is an arbitrary integer such that $(a, q) = 1$. $p_1, p_2, \dots$ denote arbitrary prime numbers. χ_0 is the principal character modulo q while χ denotes an arbitrary character modulo q. We put

$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

and

$$\psi(x,\chi) = \sum_{n \le x} \chi(n) \Lambda(n) .$$

For $k = 2, 3, \dots, x \ge 2$, $(a, q) = 1$, we denote the number of solutions of

$$p_1 p_2 \dots p_k \equiv a \pmod q , \ p_1 \le x , \ p_2 \le x, \dots , p_k \le x$$

by $f(x, a, k)$; in particular, we put

$$F(a, k) = f(q, a, k)$$

so that $F(a, k)$ denotes the number of solutions of

$$p_1 p_2 \dots p_k \equiv a \pmod q), p_1 \le q, p_2 \le q, \dots , p_k \le q .$$

Furthermore, we put

$$g(x, a, k) = \sum_{\substack{p_1 \le x, \dots, p_k \le x \\ p_1 \dots p_k \equiv a \pmod q}} \log p_1 \dots \log p_k ,$$

$$h(x, a, k) = \sum_{\substack{n_1 \le x, \dots, n_k \le x \\ n_1 \dots n_k \equiv a \pmod q}} \Lambda(n_1) \dots \Lambda(n_k) ,$$

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$$G(a, k) = g(q, a, k)$$

and

$$H(a, k) = h(q, a, k) .$$

2.

P. Erdős conjectured that for $q > q_0$ and all a satisfying $(a, q) = 1$ we have

$$F(a, 2) > 0 ,$$

i.e.,

$$p_1 p_2 \equiv a \pmod q, \; p_1 \leq q, p_2 \leq q$$

can be solved.

Unfortunately, we have not been able to prove this conjecture; in fact, this is almost certainly true, but will be hopelessly difficult to prove. The purpose of this paper is to prove some slightly weaker related results.

In particular, in Part I we prove some *conditional results*. In fact, we derive relatively sharp estimates from the following weaker form of the generalized Riemann hypothesis (it will be referred to as hypothesis $H(\theta_q, x)$):

HYPOTHESIS $H(\theta_q, x)$. θ_q, x are real numbers such that $1/2 \leq \theta_q < 1$, $x \geq 2$ and the functions $L(s, \chi)$ (where χ runs over the modulo q characters) do not vanish in the domain $\mathrm{Re}\, s > \theta_q$, $|\mathrm{Im}\, s| \leq x^{1-\theta_q}$.

The applicability of this hypothesis is based on the following lemma:

LEMMA 1. *There exists an absolute constant c_1 such that if $2 \leq q \leq x$ and the hypothesis $H(\theta_q, x)$ is true then we have*

$$|\psi(x, \chi) - E_0 x| < c_1 x^{\theta_q} (\log x)^2 \tag{1}$$

where

$$E_0 = \begin{cases} 1 & \text{for } \chi = \chi_0 \\ 0 & \text{for } \chi \neq \chi_0 . \end{cases}$$

(c_0 is independent of q, x, θ_q, χ).

In fact, (1) is identical with formula (5.10) in [2], p. 236.

In Section 4, we study the functions $F(a, 2)$, $G(a, 2)$. In Section 5, we estimate $F(a, 3)$, $G(a, 3)$. Finally, in Section 6, we seek for a possibly small function $x = x(q)$ such that for $q > q_0$ and all a satisfying $(a, q) = 1$ we have

$$f(x, a, 2) > 0 .$$

(Part II will be devoted to weaker but *unconditional* results.)

3.

We need some preliminary lemmas.

LEMMA 2. *If $a_1, a_2, \ldots, a_{q-1}$ are arbitrary complex numbers then we have*

$$\sum_{\chi} \left| \sum_{n=1}^{q-1} a_n \chi(n) \right|^2 = (q-1) \sum_{n=1}^{q-1} |a_n|^2 .$$

(We recall that q denotes a prime number.)

PROOF. See [1], p. 53.

LEMMA 3. *For $q \geq q_0$ we have*

$$\sum_{\chi} |\psi(q, \chi)|^2 < 2q^2 \log q .$$

PROOF. By the prime number theorem and Lemma 2, for large q we have

$$\sum_{\chi} |\psi(q, \chi)|^2 = \sum_{\chi} \left| \sum_{n=1}^{q-1} \chi(n) \Lambda(n) \right|^2 = (q-1) \sum_{n=1}^{q-1} (\Lambda(n))^2 <$$

$$< q \sum_{n=1}^{q-1} \Lambda(n) \log q = q \log q \sum_{n=1}^{q-1} \Lambda(n) < 2q^2 \log q .$$

LEMMA 4. *For $k = 2, 3, \ldots, q > q_0$, $x \geq q$ and $(a, q) = 1$ we have*

$$\left| g(x, a, k) - \frac{1}{q-1} \sum_{\chi} \overline{\chi}(a) \big(\psi(x, \chi)\big)^k \right| < 5kx^{k-1/2} (\log x) q^{-1} .$$

PROOF. We have

$$\frac{1}{q-1} \sum_{\chi} \overline{\chi}(a) \big(\psi(x, \chi)\big)^k = \frac{1}{q-1} \sum_{\chi} \overline{\chi}(a) \left(\sum_{n \leq x} \chi(n) \Lambda(n) \right)^k =$$

$$= \frac{1}{q-1} \sum_{\chi} \sum_{n_1 \leq x, \ldots, n_k \leq x} \overline{\chi}(a) \chi(n_1) \ldots \chi(n_k) \Lambda(n_1) \ldots \Lambda(n_k) =$$

$$= \sum_{n_1 \leq x, \ldots, n_k \leq x} \left(\frac{1}{q-1} \sum_{\chi} \overline{\chi}(a) \chi(n_1 \ldots n_k) \right) \Lambda(n_1) \ldots \Lambda(n_k) =$$

$$= \sum_{\substack{n_1 \leq x, \ldots, n_k \leq x \\ n_1 \ldots n_k \equiv a \pmod{q}}} \Lambda(n_1) \ldots \Lambda(n_k) .$$

Thus, by the prime number theorem, for large q we have

$$\left| g(x, a, k) - \frac{1}{q-1} \sum_{\chi} \bar{\chi}(a)(\psi(x, \chi))^k \right| =$$

$$= \left| \sum_{\substack{p_1 \le x, \ldots, p_k \le x \\ p_1 \ldots p_k \equiv a \pmod a}} \log p_1 \ldots \log p_k - \sum_{\substack{n_1 \le x, \ldots, n_k \le x \\ n_1 \ldots n_k \equiv a \pmod q}} \Lambda(n_1) \ldots \Lambda(n_k) \right| =$$

$$= \sum_{\substack{p_1^{\alpha_1} \le x, \ldots, p_k^{\alpha_k} \le x \\ \alpha_1 + \ldots + \alpha_k > k \\ p_1^{\alpha_1} \ldots p_k^{\alpha_k} \equiv a \pmod q}} \log p_1 \ldots \log p_k \le$$

$$\le \sum_{i=1}^{k} \sum_{\substack{p_1^{\alpha_1} \le x, \ldots, p_k^{\alpha_k} \le x \\ \alpha_i \ge 2 \\ p_1^{\alpha_1} \ldots p_{\alpha_k}^{k} \equiv a \pmod q}} (\log x)^k =$$

$$= k(\log x)^k \sum_{\substack{p_1^{\alpha_1} \le x, \ldots, p_k^{\alpha_k} \le x \\ \alpha_1 \ge 2 \\ p_1^{\alpha_1} \ldots p_k^{\alpha_k} \equiv a \pmod q}} 1 \le$$

$$\le k(\log x)^k \sum_{p_1^{\alpha_1} \le x, \alpha_1 \ge 2} \; \sum_{p_2^{\alpha_2} \le x, \ldots, p_{k-1}^{\alpha_{k-1}} \le x} \; \sum_{\substack{n_k \le x \\ p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{k-1}^{\alpha_{k-1}} n_k \equiv a \pmod q}} 1 \le$$

$$\le k(\log x)^k \sum_{p_1^{\alpha_1} \le x, \alpha_1 \ge 2} \; \sum_{p_2^{\alpha_2} \le x \ldots, p_{k-1}^{\alpha_{k-1}} \le x} \left(\left[\frac{x}{q} \right] + 1 \right) \le$$

$$\le k (\log x)^k \sum_{p_1^{\alpha_1} \le x, \alpha_1 \ge 2} \left(\frac{x}{\log x} \right)^{k-2} \frac{2x}{q} = 2kx^{k-1} (\log x)^2 q^{-1} \left(\sum_{p^2 \le x} 1 + \sum_{\substack{p^\alpha \le x \\ \alpha \ge 3}} 1 \right) =$$

$$= 2kx^{k-1} (\log x)^2 q^{-1} ((2 + o(1) x^{1/2} (\log x)^{-1} + O(x^{1/3})) < 5kx^{k-1/2} \log x \cdot q^{-1} .$$

4.

In this section, we prove the following results:

Theorem 1. *If the hypothesis $H(\theta_q, q)$ is true, then we have*

$$\sum_{a=1}^{q-1} (G(a, 2) - q)^2 < c_2 q^{1+2\theta_q} (\log q)^5 .$$

Corollary 1. *If the hypothesis $H(\theta_q, q)$ is true, then*

(2) $$F(a, 2) > 0$$

holds for all but $c_3 q^{2\theta_q - 1}(\log q)^5$ *integers a such that* $0 < a < q$. *(In particular, if the generalized Riemann hypothesis is true then* (2) *holds for all but* $c_3\,(\log q)^5$ *integers a such that* $0 < a < q$.)

PROOF of Theorem 1. By Lemmas 1 and 3, for large q we have

$$\left|\sum_{a=1}^{q-1}\left(\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(q,\chi)\big)^2 - q\right)^2\right| = \tag{3}$$

$$= \left|\sum_{a=1}^{q-1}\left(\frac{1}{q-1}\sum_{\chi_1}\overline{\chi}_1(a)\big(\psi(q,\chi_1)\big)^2\right)\left(\frac{1}{q-1}\sum_{\chi_2}\overline{\chi}_2(a)\big(\psi(q,\chi_2)\big)^2\right) - \right.$$

$$\left. - \frac{2q}{q-1}\sum_{a=1}^{q-1}\sum_{\chi}\overline{\chi}(a)\,\big(\psi(q,\chi)\big)^2 + q^2(q-1)\right| =$$

$$= \left|\frac{1}{(q-1)^2}\sum_{\chi_1}\sum_{\chi_2}\left(\sum_{a=1}^{q-1}\overline{(\chi_1\chi_2)}\,(a)\right)\big(\psi(q,\chi_1)\big)^2\,\big(\psi(q,\chi_2)\big)^2 - \right.$$

$$\left. - \frac{2q}{q-1}\sum_{\chi}\left(\sum_{a=1}^{q-1}\overline{\chi}(a)\right)\big(\psi(q,\chi)\big)^2 + q^2(q-1)\right| =$$

$$= \left|\frac{1}{(q-1)^2}\sum_{\chi}\left(\sum_{a=1}^{q-1}\chi_0(a)\right)\big(\psi(q,\chi)\big)^2\,\big(\psi(q,\overline{\chi})\big)^2 - \right.$$

$$\left. - \frac{2q}{q-1}\left(\sum_{a=1}^{q-1}\chi_0(a)\right)\big(\psi(q,\chi_0)\big)^2 + q^2(q-1)\right| =$$

$$= \left|\frac{1}{q-1}\sum_{\chi}|\psi(q,\chi)|^4 - 2q\big(\psi(q,\chi_0)\big)^2 + q^2(q-1)\right| \le$$

$$\le \left|\frac{1}{q-1}|\psi(q,\chi_0)|^4 - 2q\big(\psi(q,\chi_0)\big)^2 + q^2(q-1)\right| + \frac{1}{q-1}\sum_{\chi\ne\chi_0}|\psi(q,\chi)|^4 =$$

$$= \frac{1}{q-1}\big((\psi(q,\chi_0))^2 - q(q-1)\big)^2 + \frac{1}{q-1}\sum_{\chi\ne\chi_0}|\psi(q,\chi)|^4 =$$

$$= \frac{1}{q-1}\big((\psi(q,\chi_0) - q)(\psi(q,\chi_0) + q) + q\big)^2 + \frac{1}{q-1}\sum_{\chi\ne\chi_0}|\psi(q,\chi)|^4 \le$$

$$\le \frac{1}{q-1}\big(|\psi(q,\chi_0) - q|\,|\psi q,\chi_0) + q| + q\big)^2 + \frac{1}{q-1}\left(\max_{\chi\ne\chi_0}|\psi(q,\chi)|\right)^2\sum_{\chi}|\psi(q,\chi)|^2 <$$

$$< \frac{1}{q-1}\big(c_1 q^{\theta_q}(\log q)^2\cdot 3q + q\big)^2 + \frac{1}{q-1}\big(c_1 q^{\theta_q}(\log q)^2\big)^2 2q^2\log q <$$

$$< c_4 q^{1+2\theta_q}(\log q)^4 + c_5 q^{1+2\theta_q}(\log q)^5 < c_6 q^{1+2\theta_q}(\log q)^5\,.$$

By using Lemma 4, (3) and Cauchy's inequality, we obtain for large q that

$$(4) \qquad \left|\sum_{a=1}^{q-1}(G(a,2)-q)^2-\sum_{a=1}^{q-1}\left(\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi)^2-q)^2\right|=$$

$$=\sum_{a=1}^{q-1}\left\{\left((G(a,2)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2\right)+\left(\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2-q)\right)^2-\right.$$

$$\left.-\left(\frac{1}{q-1}\sum_{\chi}\chi(a)(\psi(q,\chi))^2-q\right)^2\right\}\Bigg|=$$

$$=\left|\sum_{a=1}^{q-1}\left\{\left(G(a,2)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2\right)^2+\right.\right.$$

$$\left.\left.+2\left(G(a,2)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2\right)\left(\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2-q\right)\right\}\right|\le$$

$$\le\sum_{a=1}^{q-1}\left|G(a,2)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2\right|^2+$$

$$+2\sum_{a=1}^{q-1}\left|G(a,2)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2\right|\left|\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2-q\right|\le$$

$$\le q(10q^{3/2}(\log q)q^{-1})^2+$$

$$+2\cdot 10q^{3/2}(\log q)q^{-1}\sum_{a=1}^{q-1}\left|\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2-q\right|\le$$

$$\le 100q^2(\log q)^2+20q^{1/2}(\log q)\left\{(q-1)\sum_{a=1}^{q-1}\left(\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2-q\right)^2\right\}^{1/2}\le$$

$$\le 100q^2(\log q)^2+20q(\log q)\left(c_6q^{1+2\theta_q}(\log q)^5\right)^{1/2}<$$

$$<100q^2(\log q)^2+c_7q^{3/2+\theta_q}(\log q)^{7/2}<c_8\,q^{3/2+\theta_q}(\log q)^{7/2}$$

(in view of $\theta_q\ge 1/2$).

(3) and (4) yield for large q that

$$\sum_{a=1}^{q-1}(G(a,2)-q)^2\le$$

$$\le\left|\sum_{a=1}^{q-1}(G(a,2)-q)^2-\sum_{a=1}^{q-1}\left(\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)(\psi(q,\chi))^2-q\right)^2\right|+$$

$$+\left|\sum_{a=1}^{q-1}\left(\frac{1}{q-1}\sum_{\chi}\chi(a)(\psi(q,\chi))^2-q\right)^2\right|<$$

$$<c_8\,q^{3/2+\theta_q}(\log q)^{7/2}+c_6\,q^{1+2\theta_q}(\log q)^5<c_9\,q^{1+2\theta_q}(\log q)^5$$

(again by $\theta_q\ge 1/2$) which completes the proof of the theorem.

PROOF of Corollary 1. Obviously, we have

$$\sum_{a=1}^{q-1}\big(G(a,2)-q\big)^2<\sum_{\substack{1\le a\le q-1\\ G(a,2)=0}}\big(G(a,2)-q\big)^2= \tag{5}$$

$$=\sum_{\substack{1\le a\le q-1\\ G(a,2)=0}}q^2=q^2\sum_{\substack{1\le a\le q-1\\ G(a,2)=0}}1=q^2\sum_{\substack{1\le a\le q-1\\ F(a,2)=0}}1\,.$$

Theorem 1 and (5) yield that

$$\sum_{\substack{1\le a\le q-1\\ F(a,2)=0}}1\le q^{-2}\sum_{a=1}^{q-1}\big(G(a,2)-q\big)^2<c_2q^{2\theta_q-1}(\log q)^5$$

which proves Corollary 1.

5.

In this section, we study $G(a, 3)$ and $F(a, 3)$.

THEOREM 2. *There exists an absolute constant c_{10} such that if $q\ge 3$ and the hypothesis $H(\theta_q, q)$ is true with*

$$\frac{1}{2}\le\theta_q<1-\frac{3}{2}\frac{\log\log q}{\log q} \tag{6}$$

then we have

$$|G(a,3)-q^2|<c_{10}q^{1+\theta_q}(\log q)^3$$

for all a stisfying $(a, q) = 1$.

COROLLARY 2. *If $\varepsilon > 0$, $q > q_0(\varepsilon)$ and the hypothesis $H(\theta_q, q)$ is true with*

$$\theta_q=1-(3+\varepsilon)\frac{\log\log q}{\log q} \tag{7}$$

then we have

$$F(a,3)>0 \tag{8}$$

for all a satisfying $(a, q) = 1$.

PROOF of Theorem 2. By Lemma 4, for $(a, q) = 1$ we have

$$\left|G(a,3)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(q,\chi)\big)^3\right|= \tag{9}$$

$$=\left|g(q,a,3)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(q,\chi)\big)^3\right|<15q^{5/2}(\log q)q^{-1}=15q^{3/2}(\log q)\,.$$

Furthermore, by using Lemmas 1 and 3, and in view of (6), we obtain that

(10)
$$\left|\frac{1}{q-1}\sum_{\chi}\bar{\chi}(a)\big(\psi(q,\chi)\big)^3 - q^2\right| =$$
$$= \left|\frac{\chi_0(a)\big(\psi(q,\chi_0)\big)^3 - q^3}{q-1} + \frac{q^2}{q-1} + \frac{1}{q-1}\sum_{\chi\neq\chi_0}\bar{\chi}(a)\big(\psi(q,\chi)\big)^3\right| \leq$$
$$\leq \frac{1}{q-1}\left|\big(\psi(q,\chi_0)\big)^3 - q^3\right| + \frac{q^2}{q-1} + \frac{1}{q-1}\sum_{\chi\neq\chi_0}|\psi(q,\chi)|^3 =$$
$$= \frac{1}{q-1}|\psi(q,\chi_0) - q|\left|\big(\psi(q,\chi_0)-q\big)^2 + 3\big(\psi(q,\chi_0)-q\big)q + 3q^2\right| + \frac{q^2}{q-1} +$$
$$+ \frac{1}{q-1}\sum_{\chi\neq\chi_0}|\psi(q,\chi)|^3 <$$
$$< \frac{2}{q}|\psi(q,\chi_0) - q|\big(|\psi(q,\chi_0)-q|^2 + 3|\psi(q,\chi_0)-q|q + 3q^2\big) + 2q +$$
$$+ \frac{2}{q}\sum_{\chi\neq\chi_0}|\psi(q,\chi)|^3 <$$
$$< \frac{2}{q}|\psi(q,\chi_0) - q|\big(3|\psi(q,\chi_0)-q|^2 + 5q^2\big) + 2q +$$
$$+ \frac{2}{q}\left(\max_{\chi\neq\chi_0}|\psi(q,\chi_0)|\right)\sum_{\chi\neq\chi_0}|\psi(q,\chi)|^2 <$$
$$< \frac{2}{q}\cdot c_1 q^{\theta_q}(\log q)^2\big(3c_1^2 q^{2\theta_q}(\log q)^4 + 5q^2\big) + 2q +$$
$$+ \frac{2}{q}\cdot c_1 q^{\theta_q}(\log q)^2 \cdot 2q^2\log q <$$
$$< c_{11}\big(q^{3\theta_q-1}(\log q)^6 + q^{1+\theta_q}(\log q)^2 + q + q^{1+\theta_q}(\log q)^3\big) <$$
$$< c_{12}\big(q^{3\theta_q-1}(\log q)^6 + q^{1+\theta_q}(\log q)^3\big) =$$
$$= c_{12}q^{1+\theta_q}(\log q)^3\big(q^{-2(1-\theta_q)}(\log q)^3 + 1\big) < 2c^{12}q^{1+\theta_q}(\log q)^3\,.$$

In view of (6), (9) and (10) yield that

$$|G(a,3) - q^2| \leq \left|G(a,3) - \frac{1}{q-1}\sum_{\chi}\bar{\chi}(a)\big(\psi(q,\chi)\big)^3\right| +$$
$$+\left|\frac{1}{q-1}\sum_{\chi}\chi(a)\big(\psi(q,\chi)\big)^3 - q^2\right| < 15q^{3/2}(\log q) + c_{13}q^{1+\theta_q}(\log q)^3 < c_{14}q^{1+\theta_q}(\log q)^3$$

which completes the proof of Theorem 2.

PROOF of Corollary 2. By Theorem 2 and in view of (7), we have

$$G(a, 3) > q^2 - c_{10}q^{1+\theta_q}(\log q)^3 = q^2\big(1 - c_{10}q^{\theta_q-1}(\log q)^3\big) =$$

$$= q^2(1 - c_{10}(\log q)^{-(3+\varepsilon)}(\log q)^3\big) > \frac{q^2}{2} > 0$$

which implies (8) (in fact, we have $F(a, 3) \gg q^2(\log q)^{-2}$).

6.

In this section, we prove the following results:

THEOREM 3. *There exists an absolute constant* c_{15} *such that if*

$$\frac{1}{2} \leq \theta_q < 1 - q^{-1/10}\ , \tag{11}$$

and writing

$$x = (c_{15}\ q)^{1/2(1-\theta_q)} \left(\frac{\log q}{1-\theta_q}\right)^{2/(1-\theta_q)}, \tag{12}$$

the hypothesis $H(\theta_q, x)$ *is true, then we have*

$$f(x, a, 2) > 0 \tag{13}$$

for all a satisfying $(a, q) = 1$.

COROLLARY 3. *There exists an absolute constant* c_{16} *such that if* $q > q_0$, *and the generalized Riemann hypothesis is true, then we have*

$$f(c_{16}q(\log q)^4, a, 2) > 0$$

for all a satisfying $(a, q) = 1$.

PROOF of Theorem 3. By Lemma 4, we have

$$\left| g(x, a, 2) - \frac{1}{q-1}\sum_{\chi} \overline{\chi}(a)\big(\psi(x, \chi)\big)^2 \right| < 10a^{3/2}(\log x)q^{-1}\,. \tag{14}$$

Furthermore, by Lemma 1 we have

$$\left| \frac{1}{q-1}\sum_{\chi} \overline{\chi}(a)\big(\psi(x, \chi)\big)^2 - \frac{x^2}{q} \right| = \tag{15}$$

$$= \left| \frac{1}{q-1} \left(\chi_0(a) \left(\psi(x, \chi_0) \right)^2 - x^2 \right) + \frac{x^2}{(q-1)q} + \frac{1}{q-1} \sum_{\chi \neq \chi_0} \bar{\chi}(a) \left(\psi(x, \chi) \right)^2 \right| \leq$$

$$\leq \frac{1}{q-1} \left| \left(\psi(x, \chi_0) \right)^2 - x^2 \right| + \frac{a^2}{(q-1)q} + \frac{1}{q-1} \sum_{\chi \neq \chi_0} |\psi(x, \chi)|^2 \leq$$

$$\leq \frac{1}{q-1} \left| \psi(x, \chi_0) - x \right| \left| \left(\psi(x, \chi_0) - x \right) + 2x \right| + \frac{2x^2}{q^2} + \frac{1}{q-1} \sum_{\chi \neq \chi_0} |\psi(x, \chi)|^2 \leq$$

$$\leq \frac{1}{q-1} \left(|\psi(x, \chi_0) - x|^2 + 2x |\psi(x, \chi_0) - x| \right) + \frac{2x^2}{q^2} + \frac{1}{q-1} \sum_{\chi \neq \chi_0} |\psi(x, \chi)|^2 <$$

$$< c_{17} \left(\frac{1}{q-1} x^{2\theta_q} (\log x)^4 + \frac{x}{q-1} x^{\theta_q} (\log x)^2 + \frac{x^2}{q^2} + \frac{1}{q-1} \sum_{\chi \neq \chi_0} x^{2\theta_q} (\log x)^4 \right) <$$

$$< c_{18} \left(x^{2\theta_q} (\log x)^4 + \frac{x^{1+\theta_q} (\log x)^2}{q} + \frac{x^2}{q^2} \right) <$$

$$< c_{19} \left(x^{\theta_q} (\log x)^2 + \frac{x}{q} \right)^2 < c_{20} \left(x^{2\theta_q} (\log x)^4 + \frac{x^2}{q^2} \right).$$

Now we are going to show that

$$c_{15} = c_{20} \tag{16}$$

can be chosen in Theorem 3.

In view of (11), for large q we have

$$\frac{(\log x)^4}{x^{2-2\theta_q}} = \frac{\left(\log \left((c_{15}\, q)^{1/2(1-\theta_q)} \left(\frac{\log q}{1-\theta_q} \right)^{2/(1-\theta_q)} \right) \right)^4}{\left((c_{15} q)^{1/2(1-\theta_q)} \left(\frac{\log q}{1-\theta_q} \right)^{2/(1-\theta_q)} \right)^{2-2\theta_q}} =$$

$$= \frac{\left(\frac{\log c_{15}\, q}{2(1-\theta_q)} + \frac{2}{1-\theta_q} \log \left(\frac{\log q}{1-\theta_q} \right) \right)^4}{c_{15} q \left(\frac{\log q}{1-\theta_q} \right)^4} =$$

$$= \frac{1}{c_{15}\, q} \left(\frac{1}{2} + \frac{\log c_{15}}{2 \log q} + \frac{1}{\log q} \log \left(\frac{\log q}{1-\theta_q} \right) \right)^4 <$$

$$< \frac{1}{c_{15} q} \left(\frac{1}{2} + \frac{\log c_{15}}{2 \log q} + \frac{2}{\log q} \log \left(\frac{\log q}{q^{-1/10}} \right) \right)^4 =$$

$$= \frac{1}{c_{15} q} \left(\frac{1}{2} + \frac{\log c_{15}}{2 \log q} + \frac{1}{5} + \frac{2 \log \log q}{\log q} \right)^4 < \frac{1}{c_{15} q} \left(\frac{1}{2} + \frac{1}{4} \right)^4 < \frac{1}{3 c_{15} q}.$$

Thus in view of (16), we obtain from (15) that for large q,

$$\left|\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(x,\chi)\big)^2-\frac{x^2}{q}\right|<c_{20}\left(x^2\frac{(\log x)^4}{x^{2-2\theta_q}}+\frac{x^2}{q^2}\right)<$$
$$<c_{20}\left(x^2\frac{1}{3c_{15}q}+\frac{x^2}{q^2}\right)=\frac{x^2}{3q}+c_{20}\frac{x^2}{q^2}<\frac{x^2}{2q}. \tag{17}$$

Formulas (14) and (17) yield for large q that

$$g(x,a,2)=\frac{x^2}{q}+\left(\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(x,\chi)\big)^2-\frac{x^2}{q}\right)+$$
$$+\left(g(x,a,2)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(x,\chi)\big)^2\right)\ge\frac{x^2}{q}-\left|\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(x,\chi)\big)^2-\frac{x^2}{q}\right|-$$
$$-\left|g(x,a,2)-\frac{1}{q-1}\sum_{\chi}\overline{\chi}(a)\big(\psi(x,\chi)\big)^2\right|>\frac{x^2}{q}-\frac{x^2}{2q}-10\frac{x^{3/2}\log x}{q}>$$
$$>\frac{x^2}{q}-\frac{x^2}{2q}-\frac{x^2}{4q}=\frac{x^2}{4q}>0$$

which implies (13) (in fact, we have

$$f(x,a,2)\gg\frac{x^2}{q(\log x)^2}\Bigg)$$

and this completes the proof of Theorem 3.

PROOF of Corollary 3. By using Theorem 3 with $\theta_q=1/2$, we obtain Corollary 3.

REFERENCES

[1] H. L. MONTGOMERY, *Topics in multiplicative number theory*, Springer, Berlin, 1971. *MR* **49:** 2616

[2] K. PRACHAR, *Primzahlverteilung*, Springer, Berlin, 1957. *MR* **19**, 393

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