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SOME COMBINATORIAL AND METRIC PROBLEMS IN GEOMETRY

P. ERDŐS

Some of these problems were discussed at our recent meeting at Siófok.

1. Let there be given n points in the plane in general position, i.e. no three on a line and no four on a circle. Let f(n) denote the largest integer so that these points determine at least f(n) distinct distances. Determine or estimate f(n) as well as possible. I have no example to show that

$$(1) \qquad f(n)/n^2 \to 0$$

and, on the other hand, I cannot prove

$$(2) \qquad f(n)/n \to \infty.$$

I feel that (1) holds, but I am less sure about (2). An old problem of mine states that if n points are in general position and $n > n_0$, then it cannot happen that the points determine n-1 distinct distances so that the ith distance occurs i times (in some order). I. Palásti and Liu have an example which shows that 7 such points are possible. $f(n) \geq n$ for $n > n_0$ would of course show that my conjecture is true.

A related problem states as follows. Let $x_1,\ldots,x_n$ be in general position. Denote by $d(x_i)$ the number of distinct distances from x_i. Trivially, $d(x_i) \geq (n-1)/3$ for every i. I am sure that there is an absolute constant $c > 0$ (i.e. independent of n and the position of the points) so that

$$(3) \qquad D(n) = \max_i d(x_i) > (1+c)n/3.$$

Is it true that there is a set $x_1,\ldots,x_n$ (in general position) for which

$$(4) \qquad D(n) < (1-c)n \ ?$$

It is rather frustrating that I got nowhere with (3) and (4). Perhaps (3) remains true if we only assume that no four of our points are on a circle or even if no circle whose center is one of the x_i's goes through more than three of the other x_i's. It would also be of interest to prove or disprove

$$(5) \qquad \sum_{i=1}^{n} d(x_i) > (1+c)n^2/3.$$

An old and no doubt very difficult problem of mine states as follows. Let $x_1,\ldots,x_n$ be n points in the plane (not necessarily in general position). Is it true that

$$(6) \qquad \max_i d(x_i) > cn/(\log n)^{1/2},$$

and perhaps even

$$(7) \qquad \sum_{i=1}^{n} d(x_i) > cn^2/(\log n)^{1/2} \ ?$$

It is very easy to show that for some $c > 0$, $\max d(x_i) > cn^{1/2}$. The only non-trivial result which points in the direction of (6) and (7) is an unpublished result of J. Beck, who proved

$$(8) \qquad \max_i d(x_i)/n^{1/2} \to \infty.$$

The proof of (8) is quite complicated.

2. Croft, Purdy and I conjectured that if n points in the plane are given then for $k \le n^{1/2}$ the number of distinct lines which contain $\ge k$ of them is less than cn^2/k^3. This conjecture was proved by Szemerédi and Trotter [1], but the best value of c is not known. Their value is almost certainly very far from being best possible. In particular, if $k = \sqrt{n}$ we obtain that the number of distinct lines which contain at least $\sqrt{n}$ of our points is less than $c\sqrt{n}$. This result is interesting since it shows the difference between finite geometries and points in the Euclidean plane. In a finite geometry of $n = p^2 + p + 1$ points one has n lines containing $p + 1 > \sqrt{n}$ points. The best value of c is not known. It is trivial and shown by the lattice points in the plane that one can give n points so that there should be $2\sqrt{n} + 2$ lines which contain $\sqrt{n}$ of our points and I thought that perhaps this is best possible, but Sah showed that one can find $(3 + o(n))\sqrt{n}$ such lines. This construction appears for the first time in the proceedings of this meeting. Perhaps it gives the best possible value of c.

3. Let there be n points in the plane, no five on a line. Denote by $g(n)$ the maximum number of distinct lines each containing four of our points. Is it true that $g(n)/n^2 \to 0$? This is an old conjecture of mine and I offer 100 dollars for a proof or disproof. Kárteszi proved that $g(n) > c\, n \log n$ is possible and Grünbaum [2] proved that $g(n) > c\, n^{3/2}$ is possible. Perhaps $g(n) < C\, n^{3/2}$, but this may be too optimistic.

4. Let $x_1, \ldots, x_n$ be n distinct points in the plane. Denote by $D(x_1,\ldots,x_n)$ the number of distinct distances determined by our points. Put

$$g(n) = \min_{x_1,\ldots,x_n} D(x_1,\ldots,x_n).$$

An old and no doubt very difficult conjecture of mine states [3]

$$(9) \qquad c_1 \cdot n/(\log n)^{1/2} < g(n) < c_2 \cdot n/(\log n)^{1/2}.$$

The upper bound is easy and is shown by the lattice points in the plane, but I offer 500 dollars for a proof or disproof of the lower bound. Of course, (9) would follow immediately from (6).

Here we are not concerned about the value of $g(n)$. Let $x_1,\ldots,x_n$ be a set of points which determines $g(n)$ distinct distances. For which n is it true that $x_1,\ldots,x_n$ is uniquely determined up to similarity? Clearly this holds for $n = 3$, the triangle must be equilateral. There is no uniqueness for $n = 4$ since $g(4) = 2$ and this can be implemented by a square or by two equilateral triangles having an edge in common. Also, $g(5) = 2$ and it seems that the regular pentagon is the only solution. A detailed proof was given by a colleague from Zagreb. (Unfortunately, I do not have his letter.) Now $g(6) = g(7) = 3$, $g(8) = 4$ and it is easy to see that there is no uniqueness here either. I thought that for $n > 5$, $g(n)$ can always be implemented in more than one way. But a colleague remarked in conversation: $g(9) = 4$ and is implemented by the regular nonagon. Is there any other way? Recently this has been decided in the positive by Gy. Hegyi, who found the following example: the six vertices of a regular hexagon, its center and the mirror images of the center with respect to two neighbouring sides. Is it true that for $n > n_0$, $g(n)$ can always be implemented in more than one way? At the moment I do not see how to attack this problem.

5. Let there be given n points in the plane. Denote by $f(n)$ the largest integer so that for every choice of the n points there should be $f(n)$ of them no two of which have distance 1. A simple example of L. and W. Moser shows that $f(n) \leq 2n/7$. L. Székely [4] proved that $f(n) > n/5$, and in fact a somewhat sharper result. Determine $f(n)$ as accurately as possible. In particular, is it true that $f(n) \geq n/4$?

A related problem states as follows: Let there be given n points in the plane and assume that 1 is the shortest distance between any two of them. Join two of them if their distance is 1. This graph is clearly planar. Denote by $g(n)$ the largest integer such that this graph always has an independent set of size $g(n)$. I thought that perhaps $g(n) = n/3$ but F. Chung and R. L. Graham, and independently J. Pach, gave a construction which shows $g(n) \leq 6n/19$. Their construction appears in Fig.1.

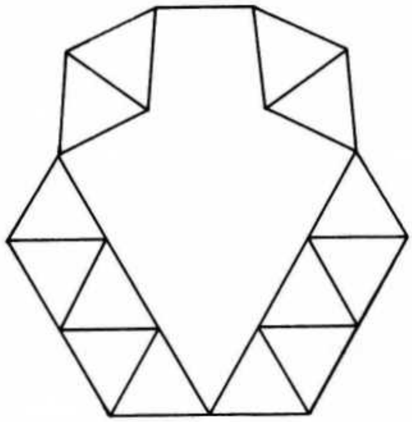

Figure 1

By the way R. Pollack [5] has a simple proof of $g(n) \geq n/4$. The determination of $g(n)$, or even $\lim g(n)/n$, is perhaps not quite easy.

6. An old theorem of Anning and myself [6] states that if S is an infinite set in the plane so that the distance between any two points of S is an integer, then S must be linear. If we only know that the distances must all be rational, S does not have to be linear but probably must have very special structure. Ulam conjectured 40 years ago that S cannot be everywhere dense and Besicovitch (independently) conjectured that the set of limit points of S cannot contain some convex n-gon for $n > n_0$.

An old problem whose origin I cannot trace states: Are there for any n, n points in general position, i.e. no three on a line and no four on a circle, so that all the $\binom{n}{2}$ distances are integers? J. Lagrange [8] found six such points, see Fig.2. Harborth just wrote me that he and Kemnitz have shown this was the example with minimal diameter, and in fact the only one with diameter at most 220.

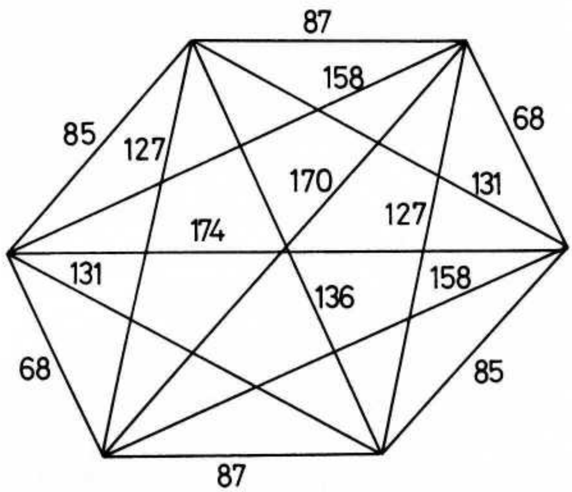

Figure 2

7. During our meeting, G. Fejes Tóth and I raised the following problem: Can one find a finite set of unit intervals in the unit square, no two of which intersect and which are maximal with respect to this property? To my surprise, Danzer found a simple example (Fig.3). This costed me 10 dollars.
Another example was found by another participant of our meeting (Fig.4 where, say, the elongation of the upper side of the lower left quadrangle passes through the lower right vertex of the square). Evidently in both examples the position of the segments can be varied. It is not clear what happens if the unit square is replaced by other regions. Also it is not clear what happens in the unit square if we insist that the only common point

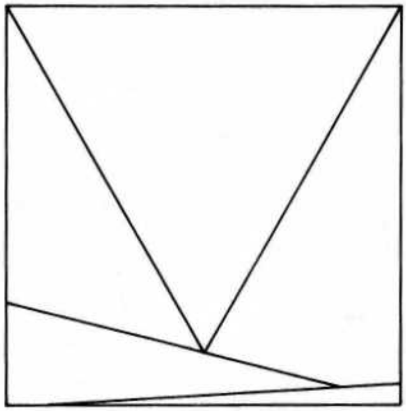

Figure 3

two of our intervals can have is their endpoint. Let R be any region and let there be given in R a maximal set of disjoint unit intervals. Can such a set ever be denumerable?

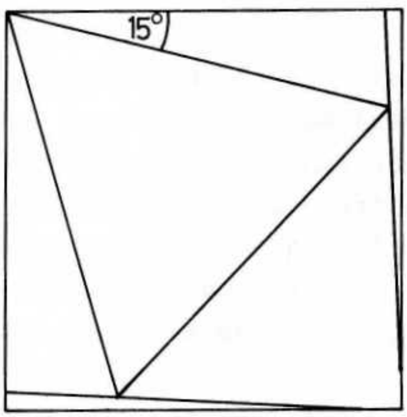

Figure 4

8. Another old problem of mine states: The vertices of a convex n-gon determine at least $[\frac{n}{2}]$ distinct distances. This conjecture was proved by Altman [7]. I further conjectured that in a convex n-gon there always is a vertex so that the number of distinct distances from this vertex is at least $[\frac{n}{2}]$. As far as I know this conjecture is still open. I also conjectured that in a convex n-gon there always is a vertex which has no three other vertices equidistant from it. This conjecture was disproved by Danzer, his example appears in Fig.5. This

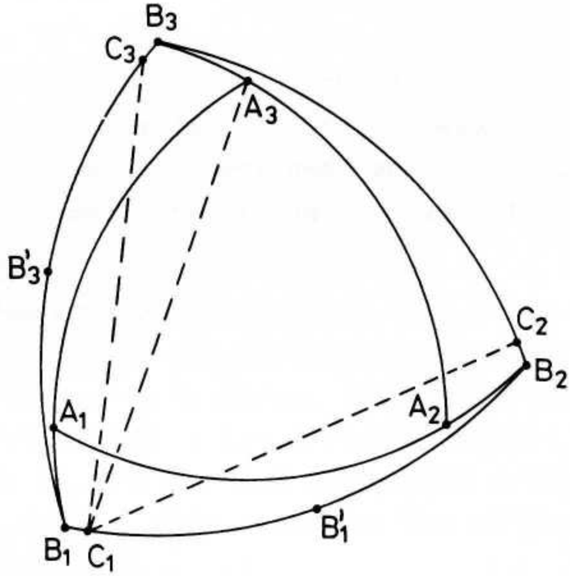

Figure 5

is a convex nonagon $A_1B_1C_1A_2B_2C_2A_3B_3C_3$ of threefold rotational symmetry, satisfying $A_1A_2 = A_1A_3 = A_1B_3$, $B_1B_2 = B_1C_2 = B_1B_3$, $C_1C_2 = C_1A_3 = C_1C_3$. It is constructed in the following way. Take a Reuleaux triangle $A_1A_2A_3$. Elongate the arc A_3A_1 beyond A_1 and choose a point B_1 on this elongation, close to A_1. Analogously we define B_2, B_3 (taking into account the threefold rotational symmetry of the figure) and we draw the Reuleaux triangle $B_1B_2B_3$. Denote B_i' the midpoint of the side B_iB_{i+1} of this Reuleaux triangle ($B_4 = B_1$). Choose a point C_1 on the arc B_1B_1' of the side B_1B_2 and analogously choose points C_2, C_3, by taking into account the rotational symmetry of the figure. For $C_1 = B_1$ we have $C_1C_3 = B_1B_3 > B_1A_3 = C_1A_3$, while for $C_1 = B_1'$ we have $C_1C_3 = B_1'B_3' < B_1'A_3 = C_1A_3$ (provided B_1A_1 is sufficiently small). Hence for some intermediate position of C_1 we will have $(C_1C_2 =)\, C_1C_3 = C_1A_3$. The nonagon constructed with this C_1 will satisfy all the requirements.

Perhaps in every convex polygon there is a vertex which does not have four other vertices equidistant from it. Finally Szemerédi conjectured that if $x_1, \ldots, x_n$ are n points no three on a line then they determine at least $[\frac{n}{2}]$ distinct distances, but he can only prove this with $[\frac{n}{3}]$.

G. Purdy and I plan to publish a survey article soon on this and related problems and perhaps later a book.

For many related problems see W. Moser, Problems in Discrete Geometry, Mimeograph Notes (1981). A new edition will soon appear in collaboration with J. Pach.

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P. ERDŐS
Mathematical Institute of the
Hungarian Academy of Sciences
H-1364 Budapest
P.O.B. 127
Hungary