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AN EXTREMAL PROBLEM FOR COMPLETE BIPARTITE GRAPHS

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Dedicated to the memory of Paul Turán

Abstract

Define $f(n, k)$ to be the largest integer q such that for every graph G of order n and size q, $\bar{G}$ contains every complete bipartite graph $K_{a,b}$ with $a+b=n-k$. We obtain (i) exact values for $f(n, 0)$ and $f(n, 1)$, (ii) upper and lower bounds for $f(n, k)$ when $k \geqq 2$ is fixed and n is large, and (iii) an upper bound for $f(n, \lfloor \varepsilon n \rfloor)$.

1. Introduction

Extremal graph theory, which was initiated by Turán in 1941 [4], is still the source of many interesting and difficult problems. The standard problem is to determine $f(n, G)$, the smallest integer q such that every graph with n vertices and q edges contains a subgraph isomorphic to G. It is striking that whereas Turán completely determined $f(n, K_m)$, there is much which is as yet unknown concerning $f(n, K_{a,b})$. In this paper, we consider a variant of the extremal problem for complete bipartite graphs. In this variant we ask how many edges must be deleted from K_n so that the resulting graph no longer contains $K_{a,b}$ for *some* pair (a, b) with $a+b=m$. Specifically, we seek to determine an extremal function $f(n, k)$ defined as follows. For $m>1$, let B_m denote the class of all graphs G such that $G \supset K_{a,b}$ for every pair (a, b) with $a+b=m$. Then for $n>k+1, f(n, k)$ is the largest integer q such that every graph G of order n and size $\binom{n}{2}-q$ is a member of B_{n-k}. In this paper we obtain exact values for $f(n, 0)$ and $f(n, 1)$, upper and lower bounds for $f(n, k)$ when $k>1$ is fixed and n is large, and an upper bound for $f(n, \lfloor \varepsilon n \rfloor)$.

2. Terminology and notation

All graphs considered in this paper will be ordinary graphs, i.e. finite, undirected graphs, without loops or multiple edges.

A graph with vertex set V and edge set E will be denoted $G(V, E)$. If $|V|=p$ and $|E|=q$, G is said to be of *order p* and *size q*. With $X, Y \subseteq V$, the set of edges in E of the form $\{x, y\}$ where $x \in X$ and $y \in Y$ will be denoted $E(X, Y)$. The complement of G will be denoted $\bar{G}$.

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The *size* of G will be given by $q(G)$. The order of the *largest connected component* will be given by $\mu(G)$ and the order of the *smallest connected component* will be given by $\eta(G)$. In particular, $\eta(G)=1$ means that G contains an isolated vertex.

Let A be a finite set. Then A^k will denote the Cartesian product $A\times A\times\ldots\times A$ with k factors and $[A]^k$ will denote the collecton of k-element subsets of A.

Where x is a real number, $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the greatest integer $\leq x$ and the least integer $\geq x$, respectively.

For any notation or terminology not explicitly mentioned in this section, we refer the reader to [1] or [2].

3. Calculation of $f(n, k)$ where k is fixed

Our starting point is the following simple observation. If G is of order n and $\mu(G)>\lceil n/2\rceil$, then $\bar{G}\not\supseteq K_{a,b}$ with $a=\lceil n/2\rceil$, $b=\lfloor n/2\rfloor$ and so $\bar{G}\notin B_n$. The opposite direction is described by the following useful lemma.

LEMMA 1. *If $G(V, E)$ is a graph of order n such that* (i) $\mu(G)\leq\{n/2\}$, (ii) $\eta(G)=1$, *and* (iii) $q(G)\leq\lfloor 2n/3\rfloor-1$, *then* $\bar{G}\in B_n$. *This result is sharp.*

PROOF. The proof is by induction on n. If $n=2$, then G is required to be empty and so the conclusion holds. Let $\mu(G)=k$. It is easy to see that the result holds if $k=1$ or 2, so we may assume that $k\geq 3$. Let $H=G-X$, where X is a component of order k. Then H is a graph of order $n-k$ and $\eta(H)=1$. Now $q(H)\leq\lfloor 2n/3\rfloor-k\leq \leq\lfloor 2(n-k)/3\rfloor-1$, the second inequality being by virtue of the fact that $k\geq 3$. Also, $\mu(H)\leq\min(k, \lfloor 2n/3\rfloor-k+1)$. If $3k\leq n$, then $k\leq\lceil(n-k)/2\rceil$ and if $3k\geq n+1$, then $\lfloor 2n/3\rfloor-k+1\leq\lceil(n-k)/2\rceil$. Hence, in all cases H satisfies (i)—(iii) and so, by the induction hypothesis, $\bar{H}\in B_{n-k}$. Since $\bar{X}$ and $\bar{H}$ are completely joined in $\bar{G}$, it follows that $\bar{G}\in B_n$.

From the remark made earlier, we know that condition (i) cannot be weakened. To see that (ii) cannot be weakened, note that if $\eta(G)>1$, then $\bar{G}\not\supseteq K_{1,n-1}$. Finally, with $n\geq 7$ set $m=\lfloor(n+1)/3\rfloor+1$, $k=\lfloor n/3\rfloor+1$, $l=n-m-k$ and consider the graph $G=T_m\cup T_k\cup\bar{K}_l$, where T_m and T_k denote arbitrary trees of orders m and k, respectively. In this case, we have $\mu(G)\leq\lceil n/2\rceil$, $\eta(G)=1$ and $q(G)=\lfloor 2n/3\rfloor$. However, $\bar{G}\not\supseteq K_{a,b}$ with $a=\lfloor 2n/3\rfloor+1$, $b=\lceil n/3\rceil-1$. This example shows that condition (iii) cannot be weakened. □

With the aid of Lemma 1, we can obtain the exact value of $f(n, k)$ in case $k=0$ or 1.

THEOREM 1. *For all* $n\geq 2, f(n, 0)=\lceil n/2\rceil-1$ *and for all* $n\geq 3, f(n, 1)=\lceil(n+1)/2\rceil$.

PROOF. With $m=\lceil n/2\rceil+1$, let $G=T_m\cup\bar{K}_{n-m}$, where T_m denotes an arbitrary tree of order m. Thus, G is a graph of order n, $q(G)=\lceil n/2\rceil$ and $\mu(G)=\lceil n/2\rceil+1$. Since $\mu(G)>\lceil n/2\rceil$, it follows that $\bar{G}\notin B_n$ and this example shows that $f(n, 0)\leq\lceil n/2\rceil-1$. To prove the inequality in the other sense, consider an arbitrary graph G of order n and size $q(G)\leq\lceil n/2\rceil-1$. Note that such a graph must satisfy (i) $\mu(G)\leq\lceil n/2\rceil$, (ii) $\eta(G)=1$, and (iii) $q(G)\leq\lfloor 2n/3\rfloor-1$. Hence, by Lemma 1, $\bar{G}\in B_n$.

With $m=\lceil(n+1)/2\rceil+1$, let $G=C_m\cup\bar{K}_{n-m}$, where C_m denotes the cycle of order m. Thus, G is a graph of order n and size $q(G)=\lceil(n+1)/2\rceil+1$. Moreover, if x is an arbitrary vertex of G, then $\mu(G-x) \geqq \lceil (n+1)/2 \rceil > \lceil (n-1)/2 \rceil$. It follows that for each x $\overline{G-x} \not\supseteq K_{a,b}$ with $a = \lceil (n-1)/2 \rceil$, $b = \lfloor (n-1)/2 \rfloor$ and so this example shows that $f(n, 1) \leqq \lceil (n+1)/2 \rceil$. To prove the inequality in the other sense, consider an arbitrary graph G of order n and size $q(G) \leqq \lceil (n+1)/2 \rceil$. Let x be a vertex of maximal degree in G, and let $H = G - x$. If x has degree $\geqq 2$, then $q(H) \leqq \lceil (n+1)/2 \rceil - 2 = \lceil (n-1)/2 \rceil - 1$. If x has degree $\leqq 1$, then G is the union of a collection of disjoint edges and so in this case as well $q(H) \leqq \lceil (n-1)/2 \rceil - 1$. Therefore, by the first part of this theorem, $\overline{H} \in B_{n-1}$ and so $\overline{G} \in B_{n-1}$. □

COROLLARY. *Let $t(n)$ denote the largest integer q such that for every graph G of order n and size q, $\overline{G}$ contains every tree of order n. For all $n \geqq 2$, $t(n) = \lceil n/2 \rceil - 1$.*

PROOF. Since each tree of order n is contained in an appropriate complete bipartite graph $K_{a,b}$ with $a+b=n$, it follows that $t(n) \geqq f(n, 0) = \lceil n/2 \rceil - 1$. On the other hand, the graph $G = (n/2)P_2$ (n even) or $G = ((n-3)/2)P_2 \cup P_3$ (n odd) is a graph of order n and size $q(G) = \lceil n/2 \rceil$ such that $\overline{G} \not\supseteq K_{1,n-1}$. (Here, mH is used to denote the graph with m components, each isomorphic to H.) This example shows that $t(n) \leqq \lceil n/2 \rceil - 1$. □

At this point, one may be tempted to conjecture that for each fixed value of k, $f(n, k) = n/2 + O(1)$, perhaps even exactly calculable as in the case of $k=0$ or $k=1$. In fact, we find that for all $k \geqq 2$, $n/2 + A\sqrt{n} < f(n, k) < n/2 + B\sqrt{n}$, where the positive numbers A and B depend only on k. Thus, there is a very striking difference between the case of $k=1$ and that of $k=2$. In order to establish the facts concerning the behavior of $f(n, k)$ when $k \geqq 2$, we shall need several preliminary results.

The following lemma uses the term *suspended path*. A path $x_0, x_1, \ldots, x_k$ in a graph G will be called suspended if its interior vertices $x_1, \ldots, x_{k-1}$ are of degree 2 in G, whereas its end vertices (x_0 and x_k) have degree $\neq 2$.

LEMMA 2. *Any tree having k vertices of degree* 1 *is the union of at most* $2k-3$ *edge-disjoint suspended paths.*

PROOF. The proof is left to the reader.

LEMMA 3. *Let T be a tree of order $n+1$ where $n \geqq 2$. There exists a vertex x such that $\mu(T-x) \leqq \lceil n/2 \rceil$. Consequently, there is a partition of the components of $T-x$ into two parts such that each part has at least $\lceil n/3 \rceil$ vertices.*

PROOF. The proof is left to the reader.

LEMMA 4. *Let $G(V, E)$ be a connected graph of order p and size $p+l-1$. With $k \geqq 2$, set $\delta = \min(\lfloor k/2 \rfloor/(4l-3), 1/4)$. Then, there exists $X \in [V]^k$ such that $\mu(G-X) \leqq \lceil (1-\delta)p \rceil$.*

PROOF. Delete l edges from G in such a way that the resulting graph H is still connected, i.e. so that H is a tree. The deleted edges determine a subtree T in the following way. First, we find those vertices which were incident in G with one of the deleted edges and so define a set A. Then, we define T to be the union of all paths in H which join pairs of vertices from A. Let A_1 denote the vertices of A which have degree 1 in T and set $A_2 = A - A_1$. According to Lemma 2, T is the union of at most $2|A_1|-3$ edge-disjoint suspended paths. The vertices of A_2 now subdivide these suspended paths into what we shall call *elementary paths*. The elementary paths may be described in the following way. The end-vertices of the elementary paths are precisely those vertices x such that either (i) $x \in A$ or (ii) $\deg(x) > 2$ in T. Suppose that there are r elementary paths $P_1, P_2, \dots, P_r$. Since $|A| \leqq 2l$, it follows that $r \leqq 2|A_1|+|A_2|-3 \leqq 4l-3$.

Note the following useful property of the construction described thus far. Suppose that x is a vertex of G and that it is not a vertex of T. Then, there is a unique path in G from x to T. If there were two such paths, then one of them would have to use one of the edges which were deleted in going from G to H. This would put x on a path in H joining two vertices from A and so force x to belong to T. In light of this property, we note that the collection of elementary paths $P_1, P_2, \dots, P_r$ may be used to define a partition $V=(V_1, V_2, \dots, V_r)$ of the vertices of G according to the following scheme. If x is an end-vertex of one or more elementary paths, it is identified with an arbitrarily chosen one of those paths. If x is an interior vertex of an elementary path, it is identified with that path. Finally, if x is a vertex of G which is not a vertex of T, let w be the other end-vertex of the unique path from x to T and identify x with the same elementary path as is w.

Now we are ready to describe and put to use the crucial properties of the elementary paths. Let u_i and v_i be the end-vertices of the i^{th} elementary path, P_i. Our construction insures that if x is any vertex of V_i other than u_i or v_i, every path from x to a vertex in $V-V_i$ contains either u_i or v_i. In other words, by deleting u_i and v_i from G, we completely disconnect the vertices of V_i from the remaining vertices of G. Without loss of generality, we may suppose that $|V_1| \geqq \dots \geqq |V_r|$. Set $m=\min(\lceil r/4 \rceil, \lfloor k/2 \rfloor)$ and consider the graph $G-X$, where $X=\{u_i, v_i, i=1, \dots, m\}$. Since $|V_1|+\dots+|V_m| \geqq mp/r \geqq \delta p$, it follows that $\mu(G-X)$ satisfies the stated bound unless $|V_1| > \lceil (1-\delta)p \rceil$. In case $|V_1| > \lceil (1-\delta)p \rceil$, set $B=V_1 \cup \{u_1, v_1\}$ and consider the tree T' spanned by the vertices of B. By Lemma 3, there exists a vertex x of this tree such that the components of $T'-x$ can be partitioned into two parts, each of cardinality at least $\lceil (|V_1|-1)/3 \rceil$. Now we may delete x and either u_1 or v_1, whichever is appropriate, and so disconnect from G a set of at least $\lfloor p/4 \rfloor$ vertices. In this case, for $X=\{x, u_1\}$ or $\{x, v_1\}$ we obtain $\mu(G-X) \leqq \lceil 3p/4 \rceil$. □

Now we are prepared to prove our theorem concerning $f(n, k)$ with $k \geqq 2$.

THEOREM 2. *Let* $k>1$ *be fixed and set* $A=\sqrt{\lfloor k/2 \rfloor/16}$ *and* $B=\sqrt{3k(k-1)/(k+1)}$. *Then, for all sufficiently large n,*

$$n/2+A\sqrt{n} < f(n, k) < n/2+B\sqrt{n}.$$

PROOF. Let $G(V, E)$ be a graph of order n and size $q=n/2+\Delta$, where $\Delta=A\sqrt{n}$. We wish to prove that there exists $X \in [V]^k$ such that $G-X$ satisfies the conditions of Lemma 1. This will establish the lower bound for $f(n, k)$. Since $\Delta=o(n)$, it follows that the number of connected components of G is at least $n-q=n/2-o(n)$. Consequently, $\eta(G) \leqq 2$. On the other hand, if $\eta(G)=2$, then $\mu(G)=o(n)$ and so by deleting just one vertex from G we obtain a graph which satisfies the conditions of Lemma 1. Hence, we now assume that $\eta(G)=1$. Since this is the case, we may assume that $\mu(G) > \lceil (n-k)/2 \rceil$, in fact $\mu(G) > \lceil (n+k)/2 \rceil$ for, otherwise, we may simply delete any k vertices from the largest component. Suppose that the largest component is of order p and size $p+l-1$. Hence, we have the bounds $p \leqq q = n/2+\Delta$ and $l \leqq q-p+1 \leqq \Delta$. With a view toward applying Lemma 4, note that if $\delta = \lfloor k/2 \rfloor/(4l-3)$ then $(1-\delta)p < (1-\lfloor k/2 \rfloor/4\Delta)(n/2+\Delta) < n/2+(\Delta^2-\lfloor k/2 \rfloor n/8)/\Delta$. Therefore, in this case and with our choice of Δ, we have $\lceil (1-\delta)p \rceil \leqq \lceil (n-k)/2 \rceil$. Certainly if $\delta = 1/4$, $\lceil (1-\delta)p \rceil \leqq \lceil (n-k)/2 \rceil$ and so the desired result follows from Lemma 4.

The upper bound is established by the following simple construction. With m chosen to be an even integer, let H be a graph of order m which is regular of degree $k+1$ and $(k+1)$-connected. An example of such a graph has vertices $0, 1, \ldots, m-1$ with two vertices i and j joined if $i-\lfloor (k+1)/2 \rfloor \leqq j \leqq i+\lfloor (k+1)/2 \rfloor \pmod m$ and, if $k+1$ is odd, i is joined to $i+m/2$ for $1 \leqq i \leqq m/2$. The fact that such a graph is, indeed, $(k+1)$-connected was proved by Harary in [3] and the proof is also given in [1, pp. 48—49]. Set $r = m(k+1)/2$ and let the edges of H be $e_1, e_2, \ldots, e_r$. For $i = 1, 2, \ldots, r$, insert a vertex y_i subdividing e_i and make y_i adjacent to $l_i - 1$ new vertices. Finally, add isolated vertices so that the resulting graph $G(V, E)$ is of order n. Thus, G is of size $q(G) = r + (l_1 + \ldots + l_r)$. Without loss of generality, we may assume that $l_1 \geqq l_2 \geqq \ldots \geqq l_r$. Now make the following choices for the parameters of G. Set $m = 2\lceil \sqrt{5kn/8(k^2-1)} \rceil$ and $l_1 = \ldots = l_k = \lceil \sqrt{5(k-1)n/8k(k+1)} \rceil \doteq l$. Then choose $l_{k+1}, \ldots, l_r$ so that $m + (l_{k+1} + \ldots + l_r) = \lceil (n-k)/2 \rceil + 1$. Let $Y = \{y_1, \ldots, y_k\}$. It is apparent that for every $X \in [V]^k$, we have $\mu(G-X) \geqq \mu(G-Y) = \lceil (n-k)/2 \rceil + 1$. Also, we have $q(G) = \lceil (n-k)/2 \rceil + 1 + kl + (k-1)m/2 < n/2 + B\sqrt{n}$ for every $\varepsilon > 0$. Since $\mu(G-X) > \{(n-k)/2\}$ for every $X \in [V]^k$, it follows that $\bar{G} \notin B_{n-k}$. This establishes the upper bound. □

4. An upper bound for $f(n, [\varepsilon n])$

At present, very little is known about $f(n, k)$ when $k \to \infty$ with n. However, the results of the preceding section suggest that $f(n, [\varepsilon n]) < \lceil (1/2+\delta)n \rceil$ where $\delta \downarrow 0$ with ε and this much can be proved without difficulty.

THEOREM 3. *Let $0 < \varepsilon < e^{-4}$ be fixed and set $\delta = \sqrt{6\varepsilon \log(1/\varepsilon)}$. For all sufficiently large values of n,*

$$f(n, \lfloor \varepsilon n \rfloor) < \lceil (1/2+\delta)n \rceil.$$

PROOF. Set $p = \lceil 1/2(1+\delta)n \rceil$, $q = \lceil (1/2+\delta)n \rceil$, $k = [\varepsilon n]$, $r = q-p$, $a = \lceil (n-k)/2 \rceil$, $b = \lfloor (n-k)/2 \rfloor$, and $c = a+p-n$. Using the probabilistic method, we shall prove the existence of a graph G of order n and size $\leqq q$ such that $\bar{G} \not\supseteq K_{a,b}$. Let $V = \{1, 2, \ldots, n\}$, $X = \{1, 2, \ldots, p\}$ and $Y = [V]^2$. The probability space used to prove the existence of G may be described as follows. Let $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 = X^p$ and $\Omega_2 = Y^r$. Each point in Ω is given probability $1/|\Omega|$. A typical point in Ω is $\omega = (\omega_1, \omega_2)$ where $\omega_1 = (x_1, \ldots, x_p)$ and $\omega_2 = (y_1, \ldots, y_r)$. Corresponding to ω there is a graph defined as follows: $\{i, j\}$ is an edge in the graph for each occurrence of $x_i = j$, $x_j = i$ or $y_k = \{i, j\}$, $k = 1, \ldots, r$. It is understood that any loops and/or extra edges which may be generated by the random method are simply not included in the graph so formed. If $\bar{G} \supseteq K_{a,b}$ then for some m, $c \leqq m \leqq a$, there are disjoint subsets of X, namely A and B with $|A|=m$ and $|B|=p-k-m$, such that $E(A,B)=\varphi$. Now for fixed A and B, consider the event $E(A,B)=\varphi$. The number of points of Ω_1 in this event is $(m+k)^m(p-m)^{p-k-m}p^k$ and the number of points of Ω_2 in this event is $\left(\binom{n}{2}-m(p-k-m)\right)^r$. Hence, we obtain the bound

$$\text{Prob}\,(\bar{G}\supseteqq K_{a,b}) \leqq \sum_{m=c}^{a}\binom{p}{m}\binom{p-m}{k}\frac{(m+k)^m(p-m)^{p-k-m}p^k}{p^p}\left(1-\frac{m(p-k-m)}{\binom{n}{2}}\right)^r.$$

Using Stirling's formula and some elementary bounds, we find that each term in the sum is bounded by

$$(1+2k/n)^n(p/k)^k\left(1-a(p-k-a)\Big/\binom{n}{2}\right)^r.$$

Substituting the values of a, k, p and r, we find that $\text{Prob}\,(\bar{G}\supseteqq K_{a,b})\to 0$ as $n\to\infty$ provided that $(1+2\varepsilon)((1+\delta)/2\varepsilon)^\varepsilon(1-(1-\varepsilon)(\delta-\varepsilon)/2)^{\delta/2}<1$. A simple calculation shows this to be the case when $0<\varepsilon<e^{-4}$ and $\delta=\sqrt{6\varepsilon\log(1/\varepsilon)}$. □

5. Additional problems and results

The bound for $f(n,\lfloor\varepsilon n\rfloor)$ provides a satisfying tie with the results for $f(n,k)$ where k is fixed; still, it leaves us with more questions than answers. Among other things, the result shows that if $F(\varepsilon)\overset{\text{def}}{=}\lim_{n\to\infty} f(n,\lfloor\varepsilon n\rfloor)/n$ exists, then $\lim_{\varepsilon\downarrow 0}F(\varepsilon)=1/2$. But, does $\lim_{n\to\infty} f(n,\lfloor\varepsilon n\rfloor)/n$ exist?

PROBLEM 1. *For* $0<x<1$, *does* $\lim_{n\to\infty} f(n,\lfloor xn\rfloor)/n$ *exist?*

By a variety of simple arguments, it is possible to prove bounds of the form $F_1(x)<f(n,\lfloor xn\rfloor)/n<F_2(x)$ which hold when $0<x<1$ is fixed and n is sufficiently large. Hence, it is at least plausible that $\lim_{n\to\infty} f(n,\lfloor xn\rfloor)/n$ exists. As an example of an upper bound for $f(n,\lfloor xn\rfloor)/n$, we give the following argument. Starting with the complete graph K_n, we wish to remove $q=\lfloor yn\rfloor$ edges $e_1,e_2,\ldots,e_q$ in such a way that all $K_{m,m}$ subgraphs with $m=\lceil(1-x)n/2\rceil$ are destroyed. Having found such a number y, we are assured that $f(n,\lfloor xn\rfloor)/n<y$. Let X_i denote the set of $K_{m,m}$ subgraphs which remain after e_i has been removed. Clearly, $|X_0|=\binom{n}{m}\binom{n-m}{m}$. At the stage of removing the edge e_{i+1} there are $|X_i|$ remaining $K_{m,m}$ subgraphs and $\binom{n}{2}-i$ remaining edges. Counting multiplicity, the remaining $K_{m,m}$ subgraphs contain $|X_i|m^2$ edges. It follows that there is an edge whose removal destroys at least $|X_i|m^2\Big/\binom{n}{2}$ of the subgraphs in X_i. By choosing such an edge for e_{i+1}, we obtain $|X_{i+1}|\leqq|X_i|\left(1-m^2\Big/\binom{n}{2}\right)$. Following such a procedure for $i=1,2,\ldots,q$, we obtain

$|X_q| < \binom{n}{m}\binom{n-m}{m}\left(1 - m^2\Big/\binom{n}{2}\right)^q$. An easy calculation using Stirling's formula allows us to conclude that if y is chosen so that $(1-(1-x)^2/2)^y < ((1-x)/2)^{1-x}x^x$ and n is sufficiently large, then $|X_q| = 0$. As $x \to 0$, the upper bound for $f(n, \lfloor xn \rfloor)/n$ that is obtained by this argument is quite inferior to the bound given in Theorem 3. The advantage of this argument is that it is applicable for all x satisfying $0 < x < 1$.

The second problem is not concerned with the calculation of $f(n, k)$, but is certainly related to the investigation described in this paper.

PROBLEM 2. *For all* $n \geqq 2$, *determine the largest integer* $m = f(n)$ *such that for every tree* T *of order* n, $\overline{T} \in B_m$.

We have obtained upper and lower bounds for $f(n)$ and these results may be published elsewhere.

Finally, we note the following generalization of the basic problem considered in this paper.

PROBLEM 3. *For* $r \geqq 2$ *and* $n \geqq k + r$, *let* $f_r(n, k)$ *denote the largest integer* q *such that for every graph* G *of order* n *and size* q, $\overline{G} \supseteq K(a_1, \ldots, a_r)$ *for every partition* $(a_1, \ldots, a_r)$ *of* $n-k$ *into* r *parts. Determine* $f_r(n, k)$.

The proofs given in this paper extend naturally and easily to the study of $f_r(n, k)$. For $r \geqq 3$, the induction argument used in the proof of Lemma 1 yields the following result.

LEMMA. *Let* $r \geqq 3$. *If* G *is a graph of order* n *such that* (i) $\mu(G) \leqq \lceil n/r \rceil$ *and* (ii) $q(G) \leqq \lfloor 2n/(r+1) \rfloor - 1$, *then* $\overline{G} \supseteq K(a_1, \ldots, a_r)$ *for every partition* $(a_1, \ldots, a_r)$ *of* n.

Now we can state the following generalizations of Theorems 1, 2 and 3. The reader will find that the proofs given earlier in the paper have been so structured that they readily yield the results now stated.

THEOREM. *For all* $r \geqq 2$ *and* $n \geqq r$, $f_r(n, 0) = \lceil n/r \rceil - 1$. *Except for certain exceptional cases,* $f_r(n, 1) = \lceil (n-1)/r \rceil + 1$ *holds for all* $r \geqq 2$ *and* $n \geqq r+1$. *The exceptional cases are* $f_3(4, 1) = 1$, $f_3(6, 1) = 2$, $f_3(8, 1) = 3$ *and, for* $r \geqq 4$, $f_r(r+1, 1) = 1$ *and* $f_r(r+2, 1) = f_r(r+3, 1) = 2$.

THEOREM. *Let* $r, k > 1$ *be fixed and set* $A = \sqrt{\lfloor k/2 \rfloor / 8r}$ *and* $B = \sqrt{6k(k-1)/((k+1)r)}$. *Then, for all sufficiently large* n,

$$n/r + A\sqrt{n} < f_r(n, k) < n/r + B\sqrt{n}.$$

THEOREM. *Let* $0 < \varepsilon < e^{-4}$ *be fixed and set* $\delta = \sqrt{r(r+1)\varepsilon \log(1/\varepsilon)}$. *For all sufficiently large values of* n,

$$f_r(n, \lfloor \varepsilon n \rfloor) < \lceil (1/r + \delta)n \rceil.$$

Exactly as in the special case of $r = 2$, the methods used in this paper provide an effective means of studying $f_r(n, k)$ only when $k \ll n$. Thus, for example, the generalization of Problem 1 to consider $f_r(n, \lfloor xn \rfloor)$, $0 < x < 1$, is an important problem about which little is known at present. With n and r fixed, $f_r(n, k)$ is defined for $0 \leqq k \leqq n-r$, and it is worth pointing out that in addition to the $k=0$ and $k=1$ cases, $f_r(n, k)$ is known exactly for $k=n-r$. We know that $f_r(n, n-r) = (n-r+t+1)s/2-1$, where $n=(r-1)s+t$, $0 \leqq t < r-1$. This is Turán's theorem.

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