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EXTREMAL THEORY AND BIPARTITE GRAPH-TREE RAMSEY NUMBERS

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For a positive integer n and graph B, $f_B(n)$ is the least integer m such that any graph of order n and minimal degree m has a copy of B. It will be show that if B is a bipartite graph with parts of order k and l $(k \le l)$, then there exists a positive constant c, such that for any tree T_n of order n and for any j $(0 \le j \le (k-1))$, the Ramsey number

$$r(T_n, B) \le n + c \cdot (f_B(n))^{j/(k-1)}$$

if $\Delta(T_n) \le (n/(k-j-1)) - (j+2) \cdot f_B(n)$. In particular, this implies $r(T_n, B)$ is bounded above by $n + o(n)$ for any tree T_n (since $f_B(n) = o(n)$ when B is a bipartite graph), and by $n + O(1)$ if the tree T_n has no vertex of large degree. For special classes of bipartite graphs, such as even cycles, sharper bounds will be proved along with examples demonstrating their sharpness. Also, applications of this to the determination of Ramsey number for arbitrary graphs and trees will be discussed.

1. Introduction

For graphs G and H, the *Ramsey number* $r(G, H)$ is the least integer N such that in any two-coloring (say with colors red and blue) of the edges of K_N, there is either a copy of G in the red subgraph or a copy of H in the blue subgraph. We investigate the Ramsey number $r(T_n, B)$, where T_n denotes a tree on n vertices and B is a bipartite graph.

Let B be a bipartite graph with parts of order k and l $(k \le l)$. Thus $B \subseteq K_{k,l}$, the complete bipartite graph. For any positive integer n, let $f_B(n)$ be the smallest positive integer m such that any graph of order n and minimal degree m contains a copy of B. The *extremal degree number* $f_B(n)$ is related to the *extremal number* $\mathrm{ext}_B(n)$, which is the minimum number of edges in a graph of order n which insures that there is a copy of B. In fact, $\mathrm{ext}_B(n) \ge n \cdot f_B(n)/2$ with the two expressions essentially the same for many graphs B. Therefore, $f_B(n) = o(n)$ for any bipartite graph, in fact, $f_B(n) \le c \cdot n^{(k-1)/k}$ for an appropriate constant c [13].

The main result that will be proved is the following, which gives an upper bound for the Ramsey number $r(T_n, B)$.

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Theorem 1. *For a fixed bipartite graph $B \subseteq K_{k,l}$ $(k \leq l)$ there exists a positive constant c such that for any j $(0 \leq j \leq k-1)$, and any tree T_n,*

$$r(T_n, B) \leq n + c \cdot (f_B(n))^{j/(k-1)},$$

when $\Delta(T_n) \leq (n/(k-j-1)) - (j+2) \cdot f_B(n)$.

For $k = 2$ or 3, the bounds given in Theorem 1 are of the right order of magnitude, and cannot be improved. In [9] it is proved that if $m = \Delta(T_n)$, then

$$r(T_n, C_4) = \max\{4, n+1, r(K_{1,m}, C_4)\}.$$

Also, $r(K_{1,m}, C_4) \leq m + c' \cdot m^{\frac{1}{2}}$, which is consistent with the degree extremal number for C_4. This verifies the sharpness of Theorem 1 for $k = l = 2$. For $k = 2$ or 3 and l arbitrary, there are similar results in [9] indicating the sharpness of Theorem 1. For $k \geq 4$, little is known about the extremal numbers of $K_{k,l}$, so it is difficult to measure how accurate the results of Theorem 1 are.

The two extreme cases of Theorem 1 ($j = k-1$ and $j = 0$) give the following two corollaries. When $j = k-1$, there is no restriction on the degree of vertices in T_n.

Corollary 2. *For a fixed bipartite graph B, there is a positive constant c such that for all trees T_n of order n,*

$$r(T_n, B) \leq n + c \cdot f_B(n).$$

The above corollary implies that for any tree T_n and bipartite graph B, $r(T_n, B) = n + o(n)$. For special classes of trees, such as those with no vertices of large degree, $r(T_n, B) = n + O(1)$. This follows from the next corollary.

Corollary 3. *For a fixed bipartite graph $B \subseteq K_{k,l}$ $(k \leq l)$ there exists a positive constant c such that for any tree T_n,*

$$r(T_n, B) \leq n + c,$$

when $\Delta(T_n) \leq (n/(k-1)) - 2 \cdot f_B(n)$.

When $B = C_4$, the constant c in Corollary 3 was shown to be 1 in [9]. It is conjectured that in fact $c = k-1$ will suffice in the general case. It is, of course, impossible to find a better constant than this, since $K_{k-1,n-1}$ contains no $K_{k,l}$ and its complement contains no connected graph of order n.

The techniques used to prove Theorem 1 can be used to obtain sharper bounds for special classes of bipartite graphs such as even cycles. Corollary 2 implies that

$$r(T_n, C_{2k}) \leq n + c \cdot n^{1/k},$$

since $f_B(n) \leq a \cdot n^{1/k}$ for $B = C_{2k}$ [2]. The next result gives an improvement of this bound when there are no vertices of extremely large degree.

Theorem 4. *For any integer $k \geqslant 2$, there exists positive constants c and d such that*

$$r(T_n, C_{2k}) \leqslant n + c$$

for any tree T_n of order n with $\Delta(T_n) \leqslant n - d \cdot n^{1/k}$.

2. Notation and terminology

Notation will generally follow that used in [1]. However, some special conventions will be used. We describe some of the special and most often used terminology.

By a *two-coloring* of a complete graph K_N we will always mean a coloring of the edges of K_N using red (R) for the first color and blue (B) for the second color. The red subgraph will be denoted by $\langle R \rangle$ and the blue subgraph by $\langle B \rangle$.

By T_m we will mean a tree of order m. A path in a graph G in which all of the interior vertices have degree two in G is called a *suspended path.* An *end-vertex* is a vertex of degree 1, and an *end-edge* is an edge incident to an end-vertex. End-edges are *independent* if no pair of them is incident. A *talon of degree m* consists of a vertex incident to m end-edges of the graph.

A bipartite graph B with parts of order k and l will be denoted by $B_{k,l}$. Thus, $B_{k,l} \subseteq K_{k,l}$. The minimum degree and maximum degree of vertices of a graph G will be denoted by $\delta(G)$ and $\Delta(G)$ respectively. The *neighborhood* of a vertex v of G will be denoted by $N_G(v)$, and the neighborhood of a set S of vertices (which is the union of the neighborhoods of the vertices of S) will be denoted by $N_G(S)$. If H is a subgraph of G, then $G - H$ is the graph obtained from G by deleting the vertices of H and any incident edges.

3. Proofs

Before proving Theorem 1 and Theorem 4, we will prove some lemmas that will handle special cases, and state some known results that will be helpful. A basis for the proof is that any large tree will have either a long suspended path, many independent end-edges, or a large degree talon. The first lemma deals with trees with long suspended paths, and the second lemma with trees with many independent end-edges.

Lemma 5. *For $l \geqslant k$ and n positive integers, let T_{n-1} be a tree with a suspended path of $l(k+l)$ vertices and T_n the tree obtained from T_{n-1} by subdividing one edge on the suspended path. If a K_{n+k-1} is two-colored such that $T_{n-1} \subseteq \langle R \rangle$, then either $T_n \subseteq \langle R \rangle$ or $B_{k,k} \subseteq \langle B \rangle$.*

Proof. Let $X = (x_1, x_2, \ldots, x_m)$ for $m = l(k+l)$ be the suspended path of T_{n-1} in $\langle R \rangle$ and let Y be the k vertices of K_{n+k-1} not in the T_{n-1}.

Assume that $T_n \nsubseteq \langle R \rangle$. Then no vertex of Y is adjacent in $\langle R \rangle$ to two consecutive vertices of X. Also, if $x_i y, x_j y \in \langle R \rangle$ for $y \in Y$, then (assuming x_i and x_j have successors along X) $x_{i+1}y, x_{j+1}y$ and $x_{i+1}x_{j+1} \in \langle B \rangle$. Therefore, if a vertex of Y is adjacent to $k+l$ vertices of X in $\langle R \rangle$, then $\langle B \rangle \supseteq K_{k+l} \supseteq K_{k,l}$. Thus, we can assume that each vertex of Y is adjacent in $\langle R \rangle$ to at most $k+l-1$ vertices of X. This implies that at least l vertices of X are adjacent to each vertex of Y in $\langle B \rangle$, which completes the proof. □

Lemma 6. *For $n > m \geq k$ and l positive integers, let T_n be a tree obtained from a tree T_{n-m} by adding m independent end-edges. Then, $r(T_n, B_{k,l}) \leq max\{r(T_{n-m}, B_{k,l}) + kl^2, n+k-1\}$.*

Proof. Let $r = \max\{r(T_{n-m}, B_{k,l}) + kl^2, n+k-1\}$ and consider a two-coloring of the graph $G = K_r$ such that $\langle B \rangle \nsupseteq B_{k,l}$, and $\langle R \rangle \nsupseteq T_n$. We will show that this leads to a contradiction.

Successively select vertex disjoint subgraphs $H_1, H_2, \ldots, H_l$ in $\langle B \rangle$ as follows: H_i is disjoint from $H_1, \ldots, H_{i-1}$ and contains a maximal number of vertices while still being isomorphic to a subgraph of $K_{k,l}$. Since $\langle B \rangle \nsupseteq K_{k,l}$, each vertex not in H_i is adjacent in $\langle R \rangle$ to at least one vertex of H_i. Let H be the union of these subgraphs. Therefore each vertex of $G - H$ is adjacent in $\langle R \rangle$ to at least one vertex in each H_i $(1 \leq i \leq l)$. By assumption there is an embedding τ of T_{n-m} into $\langle R \rangle$ such that $\tau(T_{n-m})$ is disjoint from H.

Let X be the m vertices of T_{n-m} incident to the m independent end-edges of T_n not in T_{n-m}, and let Y be the vertices of G not in $\tau(T_{n-m})$. Consider the bipartite graph L with parts $\tau(X)$ and Y induced by $\langle R \rangle$. A matching in the graph L which saturates $\tau(X)$ would imply that $\langle R \rangle \supseteq T_n$, so assume no such matching exists. Therefore, by Hall's theorem [12], there is a subset S of $\tau(X)$ such that $|N_L(S)| < |S|$. Since $V(H) \subseteq Y$, each vertex of $\tau(X)$ has degree at least l and $|S| > l$. Therefore, the vertices of S are commonly adjacent in $\langle B \rangle$ to at least $|Y| - m + 1 \geq k$ vertices of Y. This gives a $K_{k,l}$ in $\langle B \rangle$, a contradiction which completes the proof. □

The following lemma is used to verify that a tree without long suspended paths and many independent end-edges must have a large talon.

Lemma 7 [4]. *If a tree T_n does not contain any suspended path with more than s vertices, then the number of end-vertices of T_n is at least $n/(2s)$.*

The next lemma is a technical result about the extremal degree number. It is intuitively obvious and convenient for some calculations in the proof of Theorem 1.

Lemma 8. *Let $B = B_{k,l}$ and $N = n + c \cdot (f_B(n)^\alpha)$ for a constant $c > 0$ and $0 < \alpha \leq 1$. Then, for large n, $f_B(N) < 2 \cdot f_B(n)$.*

Proof. Clearly, $f_B(N) \leq f_B(n) + c \cdot (f_B(n))^\alpha$. If $\alpha < 1$, the result follows immediately. The same is true if B is a forest, for then $f_B(n)$ is bounded. Thus, we assume that $\alpha = 1$, and B contains an even cycle, so $f_B(n)$ is unbounded [13].

Let G be a graph of order N with $\delta(G) \geq 2 \cdot f_B(n)$, H a subgraph of order n, and $S = V(G - H)$. We will assume that $H \not\supseteq B$, and show that this leads to a contradiction. Since $H \not\supseteq B$, $\delta(H) < f_B(n)$ and there is an $h_1 \in H$ which is adjacent to at least $f_B(n)$ vertices of S. Assume that $h_1, \ldots, h_i$ have been shown, and let $H_i = H - \{h_1, \ldots, h_i\}$. Again, $\delta(H_i) < f_B(n)$, so there exists an $h_{i+1} \in H_i$ adjacent to at least $f_B(n) - i$ vertices of S. For $m = \lceil f_B(n) \rceil$ and $H' = \{h_1, \ldots, h_m\}$, each vertex of H' is adjacent to m vertices of S.

There are at least $\binom{m}{k}$ k-subsets of S in the neighborhood of each of the m vertices of H'. However, there are only $\binom{cm}{k}$ k-subsets of S. Thus, for m large, some k-subset is in the neighborhood of at least $m \cdot \binom{m}{k} / \binom{cm}{k} \geq l$ vertices of H'. This implies $G \supseteq B$, a contradiction which completes the proof. $\square$

The major difficulty in proving both Theorem 1 and Theorem 4 is dealing with the case of trees with large talons. The following is a greedy algorithm that will be used in embedding such trees.

Algorithm. Our objective is to describe a procedure to assist in embedding a tree T_n in $\langle R \rangle$ of a two-colored $G = K_{n+t}$ in which $\langle B \rangle \not\supseteq K_{k,l}$, and $t \geq l$. We will assume that the tree T_n contains a talon with q edges and that $\delta(\langle R \rangle) \geq n - q$.

Let v be the center of the talon, and denote the tree obtained from T_n by deleting the q edges of the talon by T_{n-q}. Let w be a vertex of maximal degree in $\langle R \rangle$, and S the vertices adjacent in $\langle B \rangle$ to w. Clearly T_{n-q} can be embedded in $\langle R \rangle$, since $\delta(\langle R \rangle) \geq n - q$, but our objective is to do this embedding in such a way that the end-vertices of the talon can also be embedded. To achieve this, we would like to use as many vertices of S as possible when we embed T_{n-q}.

Define the embedding τ of T_{n-q} as follows:

(1) Root the tree T_{n-q} at v and set $\tau(v) = w$.

(2) For $u \in V(T_{n-q})$, suppose that $\tau(u)$ has been defined, and $u_1, u_2, \ldots, u_m$ are the children of u. Select the images $\tau(u_1), \ldots, \tau(u_m)$ such that the edges $\tau(u)\tau(u_1), \ldots, \tau(u)\tau(u_m) \in \langle R \rangle$, and such that a maximum number of the vertices of $\{\tau(u_1), \ldots, \tau(u_m)\}$ are in S. If all of these vertices are not in S, label the vertex u "bad" and place it in the set D. The vertex v will always be considered a "bad" vertex.

This defines an embedding τ of T_{n-q} into $\langle R \rangle$, since $\delta(\langle R \rangle) \geq n - q$. Let S' be the vertices of S not in the image of τ. Three situations can occur.

(a) If $|S'| \leq t$, then the embedding τ can be extended to T_n, since there will be at least q vertices adjacent to w in $\langle R \rangle$ which are not in $\tau(T_{n-q})$.

(b) If $|S'| > t$, and the number of "bad" vertices $|D| \geq k$, then $\langle B \rangle \supseteq K_{k,l}$, since all edges between D and S' are in $\langle B \rangle$. This cannot occur.

(c) If $|S'| > t$, and the number of "bad" vertices $|D| < k$, then many edges of T_{n-q} will be incident to these "bad" vertices. In fact, each edge of T_{n-q} is either embedded in S, or is incident to a "bad" vertex. Therefore, at least $(n-|S-S'|)$ vertices of T_{n-q} are adjacent to the "bad" vertices. When $|S|$ is small in comparison to n, this will be used to generate vertices of large degree in T_n. In fact, T_{n-q} must contain a vertex of degree at least $(n-|S-S'|)/|D|$. This will give a contradiction under appropriate conditions that will exist when the algorithm is applied.

Proof of Theorem 1. The proof will be by induction on n, the order of the tree. An appropriate choice of c insures that the result is true for small values of n. Assume the theorem is true for all trees of order less than n, and that n is large. Let $M(j)=\lfloor c\cdot(f_B(n))^{j/(k-1)}\rfloor$ and $N=n+M(j)$, and assume that $G=K_N$ is two-colored such that $\langle R\rangle \not\supseteq T_n$ and $\langle B\rangle \not\supseteq B$. We will show that this leads to a contradiction.

The remainder of the proof will be broken into three cases:

(1) T_n has a suspended path with at least $l(k+l)+1$ vertices

(2) T_n has kl^2 independent end-edges, or

(3) T_n has a talon with at least $n/(2kl^3(k+l))$ edges.

These cases are exhaustive. If (1) does not occur, then T_n has a least $n/(2l(k+l))$ end-edges by Lemma 7. If (2) does not occur, then all these end-edges are involved in at most kl^2 talons, which gives (3).

Case (1). T_n has a suspended path with at least $l(k+l)+1$ vertices

Let T_{n-1} denote the tree obtained from T_n by decreasing the length of the suspended path by 1. By the induction assumption, $\langle R\rangle \supseteq T_{n-1}$. An appropriate choice of the constant c insures that Lemma 5 applies, which gives a contradiction in this case.

Case (2). T_n has kl^2 independent end edges

Let $m=kl^2$, and let T_{n-m} be the tree obtained from T_n by deleting m independent end-edges. Lemma 6 implies

$$\begin{aligned} r(T_n,B) &\leqslant \max\{n-m+c\cdot(f_B(n-m))^{j/(k-1)}+kl^2,\ n+k-1\}\\ &\leqslant n+c(f_B(n))^{j/(k-1)} \end{aligned}$$

for appropriate choice of c. This contradiction completes the proof of this case.

Before considering Case (3), we will make some general observations about $\langle R\rangle$ and the degree of vertices in this subgraph. Note that by the definition of $f_B(n)$, $\Delta(\langle R\rangle)\geqslant N-f_B(N)$. By Lemma 8, $f_B(N)\leqslant 2f_B(n)$, so $\Delta(\langle R\rangle)\geqslant N-2\cdot f_B(n)$. Also, the number of vertices of "small" degree in $\langle R\rangle$ is small. Consider any number p $(0<p<1)$, and let x be the number of vertices of $\langle R\rangle$ of degree less than $(1-p)n$. Each of these vertices has degree at least $\lceil pn\rceil$ in $\langle B\rangle$

and at least $\binom{\lceil pn \rceil}{k}$ subsets of cardinality k in its neighborhood. Since $\langle B \rangle \not\supseteq B$,

$$x \cdot \binom{\lceil pn \rceil}{k} \leq (l-1)\binom{N}{k}.$$

This implies that x is bounded by a function that depends only on k, l, and p, and not on n. These x vertices can be deleted without significantly changing either the number or degree of the remaining vertices (appropriately alter the constants c and p). Thus throughout the remainder of the proof we will assume that $\delta(\langle R \rangle) \geq (1-p)n$. The appropriate choice for the value of p will depend on the conditions in Case (3), which follows.

Case (3). T_n has a talon with at least $n/(2kl^3(k+l))$ edges.

Select p $(0<p<1)$ such that pn is the maximal degree of a talon in T_n. Thus certainly $pn \geq n/(2kl^3(k+l))$. Let v be the center of this talon, and T_{n-q} the tree obtained from T_n by deleting the q edges of the talon, where $q = pn$. Also, let w be a vertex of maximal degree in $\langle R \rangle$, and S the vertices adjacent to w in $\langle B \rangle$. Since $\Delta(\langle R \rangle) \geq N - 2 \cdot f_B(n)$, S has at most $2 \cdot f_B(n)$ vertices. We apply the algorithm described earlier (with $t = M(j)$). Notation used in the description of the algorithm will be used in the following discussion.

Three subcases $j = k-1$, $j = 0$, and $1 \leq j < k-1$ will be considered.

$j = k-1$

If $c \geq 2$, the algorithm yields an embedding, since $|S|$, and hence $|S'|$, is less than t and (a) of the algorithm applies. This gives a contradiction.

$j = 0$

In this case we can assume that neither (a) or (b) of the algorithm applies for otherwise we would have a contradiction. Therefore, there are at most $k-1$ "bad" vertices, and one of these vertices has degree at least $(n-|S|)/(k-1)$ by (c), which contradicts the condition on $\Delta(T_n)$ for $d \geq 2$.

$1 \leq j < k-1$

Both (a) and (b) of the algorithm give a contradiction, so we assume (c) applies. Therefore, the set of "bad" vertices D has at most $k-1$ vertices, the sum of the degrees of these vertices is at least $n-|S|$, and S' has at least $M(j)$ vertices.

Consider the $k-j$ of these vertices which have the largest degrees. The claim is that each of these vertices must have degree at least $|S|$. If not, then the sum of the degrees of the $k-j-1$ largest degree vertices would be at least $n-(j+$

1) $|S|$, and some vertex of T_n would have degree at least $(n-(j+2)|S|)/(k-j-1)$, a contradiction for $d \geq j+2$.

Let $\{v, v_1, v_2, \ldots, v_{j-k-1}\}$ be any set of $k-j$ vertices of T_{n-q} which includes v and such that each has degree at least $|S|$. Let T' be the subtree of T_{n-q} spanned by these vertices. Since the length of suspended paths and the number of independent end-edges is bounded by some function of k and l, the order of the tree T' is also bounded by a function depending only upon k and l. For any embedding τ of T' into $\langle R\rangle$ which voids S and with $\tau(v)=w$, there is a $k-j-1$ set $Y=\{\tau(v_1), \tau(v_2), \ldots, \tau(v_{j-k-1})\}$ of vertices in $V(G)-S$.

If the union of the neighborhoods in $\langle R\rangle$ of the $k-j-1$ vertices Y contain all of the vertices of S except for possibly $M(j)$, then the embedding τ can be extended to T_{n-q} using all but possibly $M(j)$ of the vertices of S. Thus clearly, τ can be extended to T_n, a contradiction. Thus, we assume that there are at least $M(j)$ vertices of S adjacent in $\langle B\rangle$ to each vertex of Y.

Since $\delta(\langle R\rangle) \geq (1-p)n$, there are many embeddings τ of T' into $\langle R\rangle$ avoiding S and with $\tau(v)=w$. In fact, the number of different $k-j-1$ subsets Y yielded by such embeddings is $b \cdot n^{k-j-1}$ for some positive constant b. Each vertex of each of these subsets Y is adjacent in $\langle B\rangle$ to at least $M(j)$ vertices of S.

Consider the bipartite graph L with the vertices in the first part being the $(k-j-1)$-subsets of $V(G)-S$, and the vertices in the second part being the k-subsets of S. If all of the edges between the $(k-j-1)$-subset and the k-subset are in $\langle B\rangle$, then the corresponding vertices in L are adjacent. If some vertex in the second part of L has degree at least $\binom{l-1}{k-j-1}+1$, then $\langle B\rangle \supseteq K_{k,l}$. Since this cannot occur, we have the following inequality (the left hand side is a lower bound on the number of edges emanating from the first part, and the right hand is an upper bound on the number of edges emanating from the second part)

$$b \cdot n^{k-j-1}\binom{M(j)}{k} \leq \left(\binom{l-1}{k-j-1}\right)\binom{|S|}{k}.$$

Using the fact that $f_B(n) \leq c''n^{(k-1)/k}$ for some constant c'', this implies that

$$(c-k)^k \leq \left(\binom{l-1}{k-j-1}\right)2^k.$$

If c is sufficiently large, this yields a final contradiction, which completes the proof of this case and the theorem. $\square$

The same techniques used in the proof of Theorem 1 apply to special cases of bipartite graphs, in particular for even cycles.

Proof of Theorem 4. The initial observations, the nature of the induction, and the proof of the first two cases are identical to the proof of Theorem 1 with C_{2k} considered as a $B_{k,k}$ bipartite graph (i.e. $l=k$). Therefore we will use precisely the same notation used in Theorem 1 with $l=k$, and assume we are at the point of beginning Case (3). Thus T_n has a talon with at least $n/(4k^5)$ edges. Recall that $f_B(n) \leq c' \cdot n^{1/k}$ for $B = C_{2k}$ [2].

Select p $(0<p<1)$ such that pn is the maximal degree of a talon in T_n. Thus certainly $pn \geq n/(4k^5)$. Let v be the center of this talon, and T_{n-m} the tree obtained from T_n by deleting the q edges of the talon, where $q = pn$. Also, let w be a vertex of maximal degree in $\langle R \rangle$, and S the vertices adjacent to w in $\langle B \rangle$. Since $\Delta(\langle R \rangle) \geq N - 2 \cdot f_B(n)$, S has at most $2 \cdot f_B(n)$ vertices.

Let T' be the tree obtained from T_{n-q} by deleting all of the vertices of degree 1. Since the length of suspended paths and the number of independent end-edges is bounded by some function of k and l, the order of the tree T' is also bounded by a function depending only upon k and l. Hence, there is an embedding τ of T' into $\langle R \rangle$ with $\tau(v) = w$ and $\tau(T')$ disjoint from S. In fact, there is such an embedding which avoids not only S but any $c''n$ vertices not in S as long as, for example, $cn \leq (1-p)n/2$.

If the embedding τ can be extended to T_{n-q} using all of the vertices of S except for possibly c, then it can clearly be extended to T_n. Thus, we assume that the embedding cannot be so extended, so there is a vertex not in S which is adjacent in $\langle B \rangle$ to at least c vertices of S. This can be repeated $c''n$ times to obtain a set A of $c''n$ vertices, each of which is adjacent in $\langle B \rangle$ to at least c vertices of S.

Consider the bipartite subgraph L of $\langle B \rangle$ induced by the parts A and S. In L, each vertex of A has degree at least c relative to S, $c \leq |S| \leq c' \cdot n^{1/k}$, and $|A| = c''n$. Therefore by a result in [11], there is a path in L of length $2k-2$ with both end-vertices in S. This path with w, which is adjacent in $\langle B \rangle$ to each vertex of S, generates a C_{2k}. This contradiction completes the proof of the theorem. □

4. Problems and comments

Two critical graphical parameters in the determination of the Ramsey number $r(S, G)$, when S is a large order sparse graph (or in particular a tree), are the order of S and the chromatic number $\chi(G)$ of G. Also, the Ramsey number $r(S, B)$ where B is a bipartite graph induced by two color classes in a $\chi(G)$ coloring of the vertices of G, appears to be an important factor in determining $r(S, G)$ [6, 8]. This is one of the motivations for working on the problems considered in this manuscript.

There are several places where the results presented could be improved; however, one is of particular interest. If T_n is a tree with only "small" degree vertices, then

$$r(T_n, B_{k,l}) = n + c$$

for a sufficiently large c. It would be nice to show that $c = k-1$ is sufficient in general. For special classes of graphs this has been verified in [4] and [9].

There are several papers dealing with the Ramsey number of a fixed graph and a sparse graph [3–5, 8]. It would be of interest to know which sparse graphs could replace the trees of Theorem 1 and Theorem 5 without altering the results.

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