JOURNAL OF MULTIVARIATE ANALYSIS 27, 169–180 (1988)

On the Area of the Circles Covered by a Random Walk

P. ERDŐS

Mathematical Institute, Budapest, Hungary

AND

P. RÉVÉSZ

Mathematical Institute, Budapest, Hungary and Technical University, Vienna, Austria

Communicated by the Editors

The area of the largest circle around the origin completely covered by a simple symmetric plane random walk is investigated.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random vectors taking values from R^2 with distribution

$$\mathbb{P}\{X_1=(0,1)\}=\mathbb{P}\{X_1=(0,-1)\}=\mathbb{P}\{X_1=(1,0)\}=\mathbb{P}\{X_1=(-1,0)\}=\tfrac{1}{4}$$

and let

$$S_0=0=(0,0) \qquad \text{and} \qquad S(n)=S_n=X_1+X_2+\cdots+X_n \qquad (n=1,2,\ldots),$$

i.e., $\{S_n\}$ is the simple symmetric random walk on the plane. Further let

$$\xi(x,n)=\#\{k: 0<k\leqslant n, S_k=x\}$$

$(n=1,2,\ldots;\ x=(i,j);\ i,j=0,\pm 1,\pm 2,\ldots)$ be the local time of the random walk. We say that the circle

$$Q(N)=\{x=(i,j): \|x\|=(i^2+j^2)^{1/2}\leqslant N\}$$

Received January 27, 1988.

AMS subject classifications: Primary 60J15; Secondary 60F15, 60G17.

Key words and phrases: random walk on the plane, strong laws, covered circle.

0047-259X/88 $3.00

is covered by the random walk in time n if

$$\xi(x, n) > 0 \qquad \text{for every} \quad x \in Q(N).$$

Let $R(n)$ be the largest integer for which $Q(R(n))$ is covered in n. We are interested in the limit properties of the random variables $R(n)$ as $n \to \infty$. This question was proposed by Erdös and Taylor [5] and they claim "we can show using the methods we have discussed above that" for any $\varepsilon > 0$

$$R(n) \geqslant \exp((\log n)^{1/2-\varepsilon}) \qquad \text{a.s.}$$

for all but finitely many n "but we have failed to get a satisfactory upper estimate and have no plausible conjecture."

This paper is devoted to the above question and some related problems.

2. A Lower Estimate of $R(n)$

In this section we prove

THEOREM 1. *For any $\varepsilon > 0$ we have*

$$R(n) \geqslant \exp\left(\frac{(\log n)^{1/2}}{(\log_2 n)^{3/4+\varepsilon}}\right) \qquad a.s.$$

for all but finitely many n where $\log_k$ is the k times iterated logarithm.

Before the proof we present a few notations and lemmas.

Let $\gamma(x, n)$ be the probability that in the first n steps the path does not pass through x i.e.

$$\gamma(x, n) = \mathbb{P}\{\xi(x, n-1) = 0\}.$$

Let $\alpha(r)$ be the probability that the random walk $\{S_n\}$ hits the circle of radius r before returning to the point $0 = (0, 0)$, i.e.,

$$\alpha(r) = \mathbb{P}\{\inf\{n : \|S_n\| \geqslant r\} < \inf\{n : n \geqslant 1, S_n = 0\}\}.$$

Further let $\beta(r, t)$ be the probability that starting from a point of the circle-ring $r \leqslant \|x\| \leqslant r+1$ the particle hits the point $0 = (0, 0)$ before hiting the circle of radius rt, i.e.,

$$\beta(r, t) = \mathbb{P}\{\inf\{n : S_{n+m} = 0\} < \inf\{n : \|S_{n+m}\| \geqslant rt\} \mid r \leqslant \|S_m\| \leqslant r+1\}.$$

Finally let

$$\delta(t) = \delta(t, r) = \mathbb{P}\{\max_{k \leq tr^2} \|S_k\| < r\}$$

and

$$\mu(x) = \mu(x, n) = \mathbb{P}\{\xi(0, n) < x \log n\}.$$

LEMMA 1. *Let* $\|x\| = \psi^{-1} n^{1/2}$ *with* $20 < \psi < n^{1/3}$. *Then*

$$\gamma(x, n) = 1 - \frac{2 \log \psi}{\log n}\left(1 + O\left(\frac{\log_2 \psi}{\log \psi}\right)\right), \tag{2.1}$$

$$\lim_{n \to \infty} \mu(x, n) = 1 - \exp(-\pi x) \tag{2.2}$$

for $0 < x < (\log n)^{3/4}$ *and the limit is approached uniformly in this range;*

$$\delta(t) = \begin{cases} 1 - \exp(-O(t^{-1})) & \textit{if} \quad t \to 0, \\ \exp(-O(t)) & \textit{if} \quad t \to \infty. \end{cases} \tag{2.3}$$

Proof. (2.1) (resp. (2.2)) are proved in Erdös and Taylor [5] cf. (2.18) (resp. Theorem 1). The proof of (2.3) is trivial.

Remark 1. (2.2) implies

$$\mathbb{P}\{\xi(0, n) = 0\} \approx \pi/\log n \tag{2.4}$$

(cf. also Dvoretzky and Erdös, [2]).

LEMMA 2. *We have*

$$\lim_{r \to \infty} \alpha(r) \log r = \pi/2. \tag{2.5}$$

Proof. Clearly we have

$$\begin{aligned}\{\inf\{n : \|S_n\| \geq r\} &> \inf\{n : n \geq 1, S_n = 0\}\} \\ &\subset \{\xi(0, r^2 \log r) > 0\} \cup \{\max_{0 \leq k \leq r^2 \log r} \|S_k\| \leq r\}.\end{aligned}$$

Since

$$\mathbb{P}\{\xi(0, r^2 \log r) = 0\} \approx \pi/2 \log r \qquad \text{by (2.4)}$$

and

$$\mathbb{P}\{\max_{0 \leq k \leq r^2 \log r} \|S_k\| \leq r\} = o(1/\log r) \qquad \text{by (2.3)},$$

we have

$$\alpha(r) \geqslant \frac{\pi + o(1)}{2 \log r}.$$

Observe also

$$\alpha(r) \leqslant \mathbb{P}\{\max_{0 \leqslant k \leqslant r^2(\log r)^{-1}} \|S_k\| \geqslant r\} + \mathbb{P}\{\xi(0, r^2(\log r)^{-1}) = 0\}.$$

Applying again (2.3) and (2.4) we obtain (2.5).

LEMMA 3. *For any $\varepsilon > 0$ and r big enough we have*

$$\beta(r, t) \leqslant (1+\varepsilon)\frac{\log_3 r}{\log r} \tag{2.6}$$

provided that $1 < t < o((\log\log r)^{\delta})$ for any $\delta > 0$.

Proof. For any $K > 0$ we have

$$\begin{aligned}\beta(r, t) &\leqslant \mathbb{P}\{\xi(0, Kr^2 + m) - \xi(0, m) \geqslant 1 \mid r \leqslant \|S_m\| \leqslant r+1\} \\ &\quad + \mathbb{P}\{\max_{m \leqslant k \leqslant m + Kr^2} \|S_k\| \leqslant rt \mid r \leqslant \|S_m\| \leqslant r+1\} = \mathrm{I} + \mathrm{II}.\end{aligned}$$

By (2.1)

$$\mathrm{I} = 1 - \gamma(x, Kr^2) \approx \frac{2 \log \psi}{\log Kr^2}$$

for any $r \leqslant \|x\| \leqslant r+1$, where $\psi = K^{1/2}r/\|x\|$ and

$$\mathrm{II} \leqslant \mathbb{P}\{\max_{0 \leqslant k \leqslant Kr^2} \|S_k\| \leqslant (t+2)r\} = \delta\left(\frac{K}{(t+2)^2}\right).$$

By choosing $K = (t+2)^2(\log\log r)^{1+\varepsilon}$ $(\varepsilon > 0)$ we obtain

$$\beta(r, t) \leqslant (1+\varepsilon)\frac{\log_3 r}{\log r}$$

for any $\varepsilon > 0$ if r is big enough and $1 < t < o((\log_2 r)^{\varepsilon})$ (for any $\varepsilon > 0$). Hence we have (2.6).

LEMMA 4. *For any $\varepsilon > 0$ and r big enough we have*

$$\beta(r, t) \geqslant 1/\varepsilon \log r \tag{2.7}$$

provided that $t \geqslant (\log\log r)^{1/2+\delta}$ for some $\delta > 0$.

Proof. For any $K>0$ we have

$$\beta(r,t)\geqslant \mathbb{P}\{\xi(0,Kr^2+m)-\xi(0,m)\geqslant 1 \mid r\leqslant \|S_m\|\leqslant r+1\}$$
$$-\mathbb{P}\{\max_{m\leqslant k\leqslant m+Kr^2}\|S_k\|\geqslant rt \mid r\leqslant\|S_m\|\leqslant r+1\}=\mathrm{I}-(1-\mathrm{II}),$$

where

$$\mathrm{I}\approx \log K/\log Kr^2$$

and

$$1-\mathrm{II}\leqslant \mathbb{P}\{\max_{0\leqslant k\leqslant Kr^2}\|S_k\|\geqslant r(t-1)\}\approx \exp\left(-O\left(\frac{(t-1)^2}{K}\right)\right)$$

provided that $K>400$ is an absolute constant and $t=t(r)\to\infty$ as $r\to\infty$. Choosing $t\geqslant(\log_2 r)^{1/2+\delta}$ with some $\delta>0$ we obtain (2.7).

In order to formulate our next lemmas we introduce some further notations. Let

$$\rho_0=0,\qquad \rho_1=\min\{k:k>0,\ S_k=0\},\ \ldots$$
$$\rho_j=\min\{k:k>\rho_{j-1},\ S_k=0\}\qquad (j=2,3,\ldots),$$
$$X_i(r)=\begin{cases}1 & \text{if } \max_{\rho_{i-1}\leqslant k\leqslant\rho_i}\|S_k\|\geqslant r,\\ 0 & \text{otherwise,}\end{cases}$$
$$Y_n(r)=\sum_{i=1}^{n}X_i(r),$$
$$Z_n(r)=Y_{\xi(0,n)}(r).$$

Clearly $Y_n(r)$ is the number of those excursions (among the first n) which are going farther than r while $Z_n(r)$ is the same number among the excursions completed before n;

$$\tau_1=\tau_1(r)=\min\{n:\ \|S_n\|\geqslant r\},$$
$$\tau_2=\tau_2(r,t)=\min\{n:n\geqslant\tau_1,\ \|S_n\|\geqslant rt\},$$
$$\tau_3=\tau_3(r,t)=\min\{n:n\geqslant\tau_2,\ \|S_n\|\leqslant r\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\tau_{2k}=\tau_{2k}(r,t)=\min\{n:n\geqslant\tau_{2k-1},\ \|S_n\|\geqslant rt\},$$
$$\tau_{2k+1}=\tau_{2k+1}(r,t)=\min\{n:n\geqslant\tau_{2k},\ \|S_n\|\leqslant r\},$$
$$\Theta_n=\Theta(n;r,t)=\max\{k:\tau_{2k+1}\leqslant n\}.$$

We say that Θ_n is the number of the $r\to rt$ excursions completed before n.

LEMMA 5. *With probability one for any $\varepsilon > 0$ we have*

$$\frac{\log n}{(\log_2 n)^{1+\varepsilon}} \leqslant \xi(0, n) \leqslant (1+\varepsilon)\, \pi(\log n) \log_3 n$$

for all but finitely many n.

Proof. See Erdös and Taylor [5, Corollary on p. 145 and Theorem 4.C].

LEMMA 6. *Let $r = r_n$ be a sequence of positive numbers with*

$$r_n \nearrow \infty, \qquad \frac{n}{\log r} \geqslant (\log n)^{2+\delta}$$

for some $\delta > 0$. Then for any $\varepsilon > 0$

$$\frac{(1-\varepsilon)\, \pi n}{2 \log r} \leqslant Y_n(r) \leqslant \frac{(1+\varepsilon)\, \pi n}{2 \log r}$$

with probability one for all but finitely many n.

Proof. It is a trivial consequence of Lemma 2.

Lemmas 5 and 6 imply

LEMMA 7. *Let $r = r_n$ be a sequence of positive numbers with*

$$r_n \nearrow \infty, \qquad \frac{\log n}{\log r} > (\log_2 n)^{3+\delta}$$

for some $\delta > 0$. Then for any $\varepsilon > 0$

$$\frac{\log n}{(\log_2 n)^{1+\varepsilon}} \frac{1}{\log r} \leqslant Z_n(r) \leqslant (1+\varepsilon) \frac{\pi^2}{2} \frac{(\log n) \log_3 n}{\log r}$$

with probability one for all but finitely many n.

LEMMA 8. *Let $r = r_n$ be a sequence of positive numbers with*

$$r_n \nearrow \infty, \qquad \frac{\log n}{\log r} > (\log_2 n)^{3+\delta}$$

for some $\delta > 0$. Then for any $\varepsilon > 0$ and for all but finitely many n we have

$$\Theta(n; r, t) \leqslant \varepsilon (\log n) \log_3 n \qquad a.s. \tag{2.8}$$

provided that

$$t \geqslant (\log_2 r)^{1/2+\delta} \qquad \textit{for some} \quad \delta > 0$$

and

$$\Theta(n; r, t) \geqslant \frac{\log n}{(\log_2 n)^{1+\varepsilon}} \frac{1}{\log_3 r} \tag{2.9}$$

provided that

$$t = o((\log_2 r)^{\delta}) \qquad \textit{for all} \quad \delta > 0.$$

Proof. (2.8) follows from Lemmas 4 and 7, (2.9) follows from Lemmas 3 and 7.

Proof of Theorem 1. Let x be an arbitrary point of the circle of radius rt, i.e., $\|x\| \leqslant rt$. Then by (2.1),

$$\begin{aligned} &\mathbb{P}\{\xi(x, \tau_{2i-1} + Kr^2t^2) - \xi(x, \tau_{2i-1}) \\ &\qquad \geqslant 1 \mid S(\tau_{2i-1}(r, t)\} \geqslant \frac{\log K}{\log Kr^2t^2} \qquad \text{a.s.,} \end{aligned} \tag{2.10}$$

provided that $400 \leqslant K \leqslant r^4t^4$. By the law of iterated logarithm one gets that

$$\tau_{(i+1)[(2K \log_2 rt)^{1/2}]}(r, t) - \tau_{i[(2K \log_2 rt)^{1/2}]}(r, t) \geqslant Kr^2t^2. \tag{2.11}$$

Consider the paths

$$\begin{aligned} &\{S_j,\ \tau_{2i[(2K \log_2 rt)^{1/2}]-1}(r, t) \leqslant j \leqslant \tau_{2i[(2K \log_2 rt)^{1/2}]-1}(r, t) + Kr^2t^2\} \\ &\qquad i = 1, 2, 3, \ldots, \quad \left[\frac{\log n}{(\log_2 n)^{1+\varepsilon}} \frac{1}{\log_3 r} \frac{1}{(2K \log_2 rt)^{1/2}}\right] \end{aligned} \tag{2.12}$$

and observe that by (2.9) all of these paths are included in the path $\{S_j, 1 \leqslant j \leqslant n\}$. (2.11) implies that the paths (2.12) are disjoint and (2.10) implies that for any x belonging to the circle of radius rt and for any i the probability that the path of (2.12) does not pass through x is less than or equal to

$$1 - \frac{\log K}{\log Kr^2t^2},$$

assuming (2.9) and (2.11).

Consequently assuming again (2.9) and (2.11), the conditional probability that the path $\{S_j, 1 \leqslant j \leqslant n\}$ does not pass through x is less than or equal to

$$\left(1-\frac{\log K}{\log Kr^2t^2}\right)^{\log n(\log_2 n)^{-1-\varepsilon}(\log_3 r)^{-1}(2K\log_2 rt)^{-1/2}}$$

$$\leqslant \exp\left(-\frac{\log K \log n}{(\log_2 n)^{1+\varepsilon}\log_3 r(2K\log_2 rt)^{1/2}\log Kr^2t^2}\right)$$

provided that

$$400 \leqslant K \leqslant r^4t^4$$

$$\frac{\log n}{\log r} > (\log_2 n)^{3+\delta} \qquad \text{for some} \quad \delta > 0,$$

$$t = o((\log_2 r)^{\delta}) \qquad \text{for all} \quad \delta > 0.$$

Choosing $K = 400$, $t = \log_3 r$, $r = \exp((\log n)^{1/2} \cdot (\log_2 n)^{-(3/4+2\varepsilon)})$, we obtain that the conditional probability that the path does not pass through x is less than or equal to

$$\exp\left(-\frac{(\log n)^{1/2}}{(\log_2 n)^{3/4-\varepsilon}}\right).$$

Consequently the probability that the path does not pass through all points of the circle of radius rt is less than or equal to

$$\exp\left(2\frac{(\log n)^{1/2}}{(\log_2 n)^{3/4+2\varepsilon}}\right)\exp\left(-\frac{(\log n)^{1/2}}{(\log_2 n)^{3/4-\varepsilon}}\right),$$

which easily proves Theorem 1.

3. Circles Covered with Positive Density

Theorem 1 gave a lower estimate of $R(n)$. Unfortunately we do not have any non-trivial upper estimation. The result of Theorem 2 suggests that $R(n)$ can be much bigger. In order to formulate our result, introduce the following notations

$$I(x, n) = \begin{cases} 1 & \text{if} \quad \xi(x, n) > 0, \\ 0 & \text{if} \quad \xi(x, n) = 0, \end{cases} \tag{3.1}$$

$$K(N, n) = (N^2\pi)^{-1} \sum_{x \in Q(N)} I(x, n);$$

i.e., $K(N, n)$ is the density of the points of $Q(N)$ covered by the random walk $\{S_k, 0 \leqslant k \leqslant n\}$. We prove

THEOREM 2. *For any* $0<\alpha<1/2$

$$\limsup_{n\to\infty} K(n^\alpha, n) \geqslant (1-2\alpha)[1-((1-\alpha)^{-1}-1)^{1/2}] \qquad a.s.$$

The proof is based on the following two lemmas.

LEMMA 9. *Let* $20<\|x\|<n^{1/3}$. *Then*

$$\gamma(x,n)=\frac{2\log\|x\|}{\log n}\left(1+O\left(\frac{\log_3\|x\|}{\log\|x\|}\right)\right). \tag{3.2}$$

Proof. See Erdös and Taylor [5, (2.16)].

LEMMA 10. *We have*

$$\mathbb{E}(I(x,n)\,I(y,n)) \leqslant \frac{(1-\gamma(x-y,n))(1-(\gamma(x,n)+\gamma(y,n))/2)}{1-\gamma(x-y,n)/2}.$$

Proof. For any lattice point z let

$$v_z=\min\{k: k>0,\ S_k=z\}.$$

Then we have

$$\begin{aligned}
&\mathbb{E}(I(x,n)\,I(y,n))\\
&\quad=\mathbb{P}(I(x,n)=1,\ I(y,n)=1)\\
&\quad=\sum_{k=0}^{n}\mathbb{P}\{I(x,n)=1,\ I(y,n)=1 \mid v_x=k<v_y\}\,\mathbb{P}\{v_x=k<v_y\}\\
&\qquad+\sum_{k=0}^{n}\mathbb{P}\{I(x,n)=1,\ I(y,n)=1 \mid v_y=k<v_x\}\,\mathbb{P}\{v_y=k<v_x\}\\
&\quad=\sum_{k=0}^{n}\mathbb{P}\{I(y,n)=1 \mid v_x=k<v_y\}\,\mathbb{P}\{v_x=k<v_y\}\\
&\qquad+\sum_{k=0}^{n}\mathbb{P}\{I(x,n)=1 \mid v_y=k<v_x\}\,\mathbb{P}\{v_y=k<v_x\}\\
&\quad=\sum_{k=0}^{n}\mathbb{P}\{I(y-x,n-k)=1\}\,\mathbb{P}\{v_x=k<v_y\}\\
&\qquad+\sum_{k=0}^{n}\mathbb{P}\{I(x-y,n-k)=1\}\,\mathbb{P}\{v_y=k<v_x\}\\
&\quad\leqslant\mathbb{P}\{I(x-y,n)=1\}\,\mathbb{P}\left\{\sum_{k=0}^{n}\{\{v_x=k<v_y\}+\{v_y=k<v_x\}\}\right\}\\
&\quad=\mathbb{P}\{I(x-y,n)=1\}\,\mathbb{P}\{I(x,n)=1 \text{ or } I(y,n)=1\}\\
&\quad=\mathbb{P}\{I(x-y,n)=1\}[\mathbb{P}(I(x,n)=1)\\
&\qquad+\mathbb{P}(I(y,n)=1)-\mathbb{P}(I(x,n)=1,\ I(y,n)=1)].
\end{aligned}$$

Hence

$$\mathbb{P}(I(x,n)=1, I(y,n)=1) \leqslant \frac{\mathbb{P}(I(x-y,n)=1)[\mathbb{P}(I(x,n)=1)+\mathbb{P}(I(y,n)=1]}{\mathbb{P}(I(x-y,n)=1)+1}$$

and we have the lemma.

Proof of Theorem 2. Apply Lemmas 1 (resp. Lemmas 9 and 10) with

$$\frac{n^\alpha}{\log n} \leqslant \|x\|, \|y\|; \|x-y\| \leqslant n^\alpha \qquad (0<\alpha<\tfrac{1}{2}).$$

We get

$$\mathbb{E}(I(x,n)\, I(y,n)) \leqslant \frac{(1-2\alpha)^2}{(1-\alpha)} \qquad (n \text{ big enough})$$

and

$$\mathbb{E}I(x,n) \approx 1-2\alpha.$$

A simple calculation gives

$$\mathbb{E}(K(n^\alpha,n)-\mathbb{E}K(n^\alpha,n))^2 \leqslant \frac{(1-2\alpha)^2}{1-\alpha}-(1-2\alpha)^2$$

and

$$\mathbb{E}K(n^\alpha,n) \approx 1-2\alpha.$$

Hence by the Chebishev inequality we have

$$\mathbb{P}\{K(n^\alpha,n)>(1-\varepsilon)(1-2\alpha)[1-((1-\alpha)^{-1}-1)^{1/2}]\} \geqslant \delta_\varepsilon>0$$

for any $\varepsilon>0$ if n is big enough. Hence we have Theorem 2.

4. Some Further Problems

In Section 2 we have studied the area of the largest circle around the origin covered by the random walk $\{S_k,\ k\leqslant n\}$. The analog problem is clearly meaningless since in R^d $(d\geqslant 3)$ the largest covered sphere is finite with probability one. However, one can ask in any dimension about the radius of the largest sphere (not surely around the origin) covered by the random walk in time n. Formally speaking, let

$$Q(N, u) = \{x : \|x - u\| \leqslant N\}$$

and $R^*(n)$ be the largest integer for which there exists a r.v. $u = u(n)$ such that

$$\xi(x, n) \geqslant 1 \qquad \text{if} \quad x \in Q(R^*(n), u).$$

It is trivial to see that in R^d

$$R^*(n) \geqslant \text{Const}(\log n)^{1/d}.$$

However, we do not have any non-trivial estimate.

In case $d = 2$ clearly $R^*(n) \geqslant R(n)$. We conjecture that $R^*(n)$ will not be larger than $R(n)$, but cannot settle this question. In fact this question is somewhat related to the problem of favourite values (cf. Bass and Griffin [1], Erdös and Révész [3], (1984), Erdös and Révész [4]).

The analogous question in the case of spheres covered with positive density can be also raised.

We also propose to investigate the area T_n of the smallest convex hull of the path $\{S_k, k \leqslant n\}$. Here we mention only a trivial result,

$$T_n \leqslant 2\pi n \log_2 n \qquad \text{a.s.} \tag{4.1}$$

for all but finitely many n,

$$T_n \geqslant \varepsilon n \log_2 n \qquad \text{a.s.} \qquad \text{i.o.} \tag{4.2}$$

with some suitable $\varepsilon > 0$.

Proof. (4.1) is a trivial consequence of the law of iterated logarithm. Let $S_n = (U_n, V_n)$. Then for any $\varepsilon > 0$

$$\mathbb{P}\{|V_n| \leqslant \varepsilon\sqrt{n}, U_n \geqslant \varepsilon(n \log_2 n)^{1/2}\} = O((\log n)^{-\varepsilon^2/2}).$$

Consider the first crossing of the path after n with the positive y axis assuming that $|V_n| \leqslant \varepsilon\sqrt{n}$, $U_n \geqslant \varepsilon(n \log_2 n)^{1/2}$. Then with a positive probability this crossing point will be farther from the origin than $(\varepsilon/2)(n \log_2 n)^{1/2}$. The time needed to get this point will not be more than n with probability $O((\log n)^{-\varepsilon})$. Hence the path $\{S_k, k \leqslant 2n\}$ meets the points $(\varepsilon(n \log_2 n)^{1/2}, 0)$ and $(0, (\varepsilon/2)(n \log_2 n)^{1/2})$ with probability $O((\log n)^{-2\varepsilon})$. Having this result, (4.2) can be obtained with the usual methods.

Note added in proof. The following result can be obtained trivially:

THEOREM 2*. *For any* $0 < \alpha < 1/2$

$$\limsup_{n \to \infty} K(n^{\alpha}, n) \geqslant 1 - 2\alpha. \qquad a.s.$$

REFERENCES

[1] BASS, R. F., AND GRIFFIN, P. S. (1985). The most visited site of Brownian motion and simple random walk. *Z. Wahrsch. Verw. Gebiete* **70** 417–436.

[2] DVORETZKY, A., AND ERDŐS, P. (1950). Some problems in random walk in space. In *Proceedings, Second Berkeley Symposium*, pp. 353–368.

[3] ERDŐS, P., AND RÉVÉSZ, P. (1984). On the favourite points of a random walk. In Mathematical structures—Computational mathematics— Mathematical modelling Vol. 2. Sofia.

[4] ERDŐS, P., AND RÉVÉSZ, P. (1987). Problems and results of random walk. In *Mathematical Statistics and Probability Theory*, Vol. B (P. Bauer, F. Konecny, and W. Wertz, Eds.), Reidel, Dordrecht.

[5] ERDŐS, P., AND TAYLOR, S. J. (1960). Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar.* **11** 137–162.