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PROBLEMS AND RESULTS IN COMBINATORIAL ANALYSIS AND GRAPH THEORY

Paul ERDŐS
Dept. of Mathematics, Hungarian Academy of Sciences, Realtanoda u. 13–15, Budapest H-1053, Hungary

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I wrote many papers with this and similar titles. In my lecture I stated several of my old solved and unsolved problems some of which have already been published elsewhere. To avoid overlap as much as possible, I state here only relatively new problems.

First I state two recent problems of Nesetril and myself.

1. Let G be a graph each vertex of which has degree not exceeding n. It is true that if our G has more than $5n^2/4$ edges then G contains two strongly independent edges? (i.e. two edges, which are vertex disjoint and for which the subgraph of G induced by the vertices of these two edges contains only these two edges).

Very recently Fan Chung and Trotter and independently and simultaneously Gyárfás and Tuza proved this conjecture. The proofs are quite complicated. It is easy to see that the result is best possible.

We also formulated the following much more difficult and interesting Vizing type conjecture: Let G be a graph each vertex of which has degree not exceeding n. Is it then true that G is the union of at most $5n^2/4$ sets of strongly independent edges? If this conjecture fails one can try to determine the smallest $f(n)$ so that every G each vertex of which has degree $\leqslant n$ is the union of $f(n)$ sets of strongly independent edges. $f(n) < 2n^2$ is easy.

One could perhaps try to determine the smallest integer $h_r(n)$ so that every G of $h_r(n)$ edges each vertex of which has degree $\leqslant n$ contains two edges so that the shortest path joining these edges has length $\geqslant r$. The order of magnitude of $h_r(n)$ is easily seen to be n^{r+1} but the exact value of $h_r(n)$ is unknown. This problem seems to be interesting only if there is a nice expression for $h_r(n)$.

Our second problem states as follows: Let $G(n)$ be a graph of n vertices $x_1, \ldots, x_n$. A subset $x_{i_1}, \ldots, x_{i_r}$ is said to be a minimal cut if the omission of these vertices disconnects $G(n)$, but no subset $x_{i_1}, \ldots, x_{i_r}$ disconnects $G(n)$. Denote by $c(n)$ the maximal number of minimal cuts a $G(n)$ can have. Seymour observed $c(3m+2) \geqslant 3^m$. To see this let $G(3m+2)$ have the vertices x, y and there be m independent paths of length 4 joining x and y. Perhaps $c(3m+2) = 3^m$, we could not even prove that $c(n)^{1/n} \to \alpha < 2$.

2. Gallai conjectured more than a year ago that if $G(n)$ is a graph which contains no wheel (i.e. a cycle and a vertex joined to all the vertices of our cycle) then $G(n)$ contains at most $\frac{1}{8}n^2$ triangles. It is easy to see that this conjecture if true is best possible. Let $|A| = \frac{1}{2}n$, $|B| = [\frac{1}{2}(n+1)]$, join every vertex of A to every vertex of B and add in B a matching. It is well known and not hard to see that every graph of $\frac{1}{4}n^2 + \frac{1}{4}n + 1$ edges contains a wheel (i.e. our graph is the largest graph which has no wheel). Unfortunately this does not seem to help with Gallai's conjecture.

3. Recently Gallai and I posed the following problem: Denote by $h(n)$ the smallest integer so that every $G(n)$ has a set of $\leqslant h(n)$ vertices $x_1, \ldots, x_t$, for which every clique of $G(n)$ contains at least one of these x_i's. It is easy to see that $h(n) \leqslant n - \sqrt{n}$. We conjecture that $h(n)$ equals to the smallest integer for which every graph of n vertices which has no triangles has a set of at least $n - h(n)$ independent vertices. To convince the reader that our conjecture is not unreasonable consider the set of all graphs on n vertices which have no triangle. Such a graph must have at least $n - h(n)$ independent vertices and there is such a graph $G_1(n)$ which has no triangle and which has exactly $n - h(n)$ independent vertices. Thus to represent all cliques (i.e. in this case all edges) of our graph we need $h(n)$ vertices (namely the complement of our largest independent set). Thus it was perhaps not unreasonble to assume that $h(n)$ vertices will always suffice to represent all cliques of our graph. We made no progress with this conjecture which is perhaps completely wrongheaded. We could not make any progress even if we assumed that our $G(n)$ has no $K(4)$. In this case we only would have to represent all $K(3)$'s of $G(n)$ and all the edges not contained in a $K(3)$.

Gallai further conjectured that if $G(n)$ is a chordal graph (i.e. all cycles C_n, $n > 3$ have a diagonal) then all cliques can be represented by $[\frac{1}{2}n]$ vertices. This conjecture was indeed proved by Aigner, Andreae and Tuza.

4. The problem of Gallai and myself naturally leads to the problem of Ramsey numbers. Many papers on these questions were published and to avoid repetition I state here only a few of them and will give an admittedly incomplete list of references.

Denote by $r(u, v)$ the smallest integer so that every graph on $r(u, v)$ vertices either contains a complete graph of u vertices or an independent set of v vertices. It is more usual to use the following (equivalent) definition: $r(u, v)$ is the smallest integer so that if we color the edges of $K(r(u, v))$ (i.e. the complete graph of $r(u, v)$ vertices) by two colors I and II then there is either a $K(u)$ all whose edges have color I or a $K(v)$ all whose edges have color II. $r(n, n)$ is the diagonal Ramsey number, it is the smallest integer for which the every complete graph of $r(n, n)$ vertices whose edges are coloured by two colors always contains a monochromatic $K(n)$. $r(3, 3) = 6$, $r(4, 4) = 18$ (this is an old result of Greenwood and Gleason) and $r(5, 5)$ is unknown. The best current bounds are

$$c_1 n 2^{n/2} < r(n, n) < \binom{2n}{n} \Big/ (\log n)^{\varepsilon}. \tag{1}$$

I proved the lower bound in (1) by probabilistic methods. The value of the constant was improved by Joel Spencer. The upper bound in (1) was recently obtained by Rödl and is not yet published. I offer 100 dollars for a proof that

$$\lim_{n\to\infty} r(n, n)^{1/n} = c \tag{2}$$

exists and I offer 10 000 dollars for a disproof. I am of course sure that (2) holds. I offer 250 dollars for the determination of c. $\sqrt{2} \leqslant c \leqslant 4$ follows from (1), perhaps $c = 2$? Let us now give a very short discussion of the non-diagonal Ramsey numbers. We have

$$\frac{c_1 n^2}{(\log n)^2} < r(3, n) < \frac{c_2 n^2}{\log n}. \tag{3}$$

The lower bound in (3) is due to me, the upper bound is due to Ajtai, Komlós and Szemerédi, who improved by a factor $\log \log n$ the previous result of Graver and Yackel. It is perhaps not hopeless to try to get an asymptotic formula for $r(3, n)$.

It would be reasonable to guess that for every fixed k and $\varepsilon > 0$ if $n > n_0(\varepsilon, k)$

$$r(k, n) > n^{k-1-\varepsilon} \tag{4}$$

but the proof of (4) presented so far unsurmountable difficulties, even for $k = 4$. At first I thought that the difficulties are only technical and the probability method will give (4), but perhaps I was too optimistic.

The best constructive lower bound for $r(n, n)$ is due to Peter Frankl, who proved

$$r(n, n) > \exp\left(\frac{c(\log n)^2}{\log \log n}\right).$$

I offer 100 dollars for a constructive proof of $r(n, n) > (1 + c)^n$. I am afraid that there are easier methods of earning 100 dollars.

Several of us tried to prove simple inequalities between Ramsey numbers. We all failed so far. The main difficulty is perhaps the lack of constructive methods. Here is a sample which shows our ignorance:

Is it true that

$$r(n + 1, n) - r(n, n) > cn^2. \tag{5}$$

'Clearly' (?).

$$\lim r(n + 1, n)/r(n, n) = C^{\frac{1}{2}} \quad \text{where} \quad r(n, n)^{1/n} \to C. \tag{6}$$

(6) seems quite hopeless at present. V.T. Sós and I failed to prove

$$\frac{r(3, n+1) - r(3, n)}{n} \to 0 \quad \text{and} \quad r(3, n+1) - r(3, n) \to \infty. \tag{7}$$

The second inequality in (7) should be perhaps easier than the first. Simonovits and I tried unsuccessfully to prove that for every $k \geqslant 4$

$$\lim_{n \to \infty} r(k+1, n)/r(k, n) = \infty \tag{8}$$

(4) is easy for $k = 3$.

I just mention one of two problems on generalized Ramsey numbers. $r(n; \mathcal{G})$ is the smallest integer t, so that if we color the edges of $K(t)$ by two colors then either color I contains a $K(n)$ or color II contains G. It is particularly frustrating that (C_4 is a cycle of length four)

$$\lim_{n \to \infty} r(n, K(3))/r(n, C_4) = \infty \tag{9}$$

has not yet been proved. (9) is an old conjecture of mine. I in fact conjectured that the following much sharper (and much more doubtful) result holds:

$$r(n; C_4) < n^{2-\varepsilon} \text{ for some } \varepsilon > 0 \quad \text{and} \quad n > n_0(\varepsilon).$$

Szemerédi proved (unpublished)

$$r(n; C_4) < cn^2/(\log n)^2. \tag{10}$$

(10) in view of (3) 'nearly' proves (9). To end this chapter I state an old and nearly forgotten conjecture of Bondy and myself: Let n be odd. Color the edge of a $K(4n-3)$ by three colors. Then there always is a monochromatic C_n. The analogous conjecture for two colors was proved by V. Rosta and Faudree and Schelp.

Several excellent survey papers on Ramsey numbers were written by Burr and Rosta, see also a forthcoming book on this subject by Burr, Faudree and Schelp. Faudree, Rousseau, Schelp and I have many papers on this subject.

References

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[2] P. Erdős, On the combinatorial problems which I would most like to see solved, Combinatorica 1 (1981), 25–42. All references can be found in one of these papers.

[3] R.L. Graham and V. Rödl, Numbers in Ramsey theory, Proc. 11th British Comb. Conf. London (1987) London Math. Soc. Lecture Notes.

[4] R.L. Graham, B. Rothschild and T. Spencer, Ramsey theory (John Wiley, New York, 1980).

5. During my last visit to Memphis State University Faudreee, Rousseau, Schelp and I came across the following nice problem which to our surprise is

perhaps difficult. (The problem came up in connection of our work on generalised Ramsey theory, but as such is quite independent of it.) Is it true that there is an absolute constant $c>0$ so that every $\mathcal{G}(n;2n-1)$ has a subgraph $G(m)$, $m<n(1-c)$ so that every vertex of $G(m)$ has degree ≥ 3? Faudree could prove this with $m \leq n-c\sqrt{n}$ instead of $n(1-c)$. C_{n-1} and an nth vertex joined to every vertex of our C_{n-1} shows that $2n-1$ cannot be replaced by $2n-2$.

Pósa and I proved that every $\mathcal{G}(n;n+k)$ contains a cycle of size not exceeding $c_1 n \log k/k$ and apart from the value of c_1 this is best possible. It might be of some interest to try to obtain the exact size $n \cdot g(k)$ of this cycle for small values of k, for example $g(1)=\frac{2}{3}$, $g(5)=\frac{1}{3}$. I believe we determined $g(k)$ for $k \leq 5$, for larger values of k the exact determination of $g(k)$ gets more and more laborious and tricky. A cycle can of course be considered as a subgraph of degree 2, but perhaps our old result with Pósa throws no light on our conjecture, since it is not difficult to prove that for every c_1 there is a c_2 so that there is a $\mathcal{G}(n;c_1 n)$ for which every subgraph $\mathcal{G}(m)$ each vertex of which has degree ≥ 3 has more than $c_2 n$ vertices. The existence of a such a graph follows easily by the probability method, but a direct construction will perhaps also be easy. We have not determined the exact dependence of c_2 on c_1.

Reference

[1] P. Erdős and L. Pósa, On the maximal number of disjoint circuits of a graph, Publ. Math. Debrecen 9 (1962) 3–12.

6. Several years ago Sauer and I asked the following question: Let $f_k(n)$ be the smallest integer for which every $G(n;f_k(n))$ contains a regular subgraph of degree k. Trivially $f_2(n)=n$, but we could not get any non-trivial results for $f_k(n)$ for $k=3$. In particular we could not prove $f_3(n)/n \to \infty$ and $f_3(n)<n^{1+\varepsilon}$. A few months ago Pyber proved

$$f_k(n) < c_2 k^2 n \log n \tag{1}$$

Pyber, Rödl and Szemerédi proved

$$cn \log\log n < f_3(n). \tag{1'}$$

Their proof of both the upper and lower bound of (1) is ingenious. It would be nice to improve (1) further and get an asymptotic formula for $f_3(n)$ and generally, $f_k(n)$.

Szemerédi once asked: Denote by $F_k(n)$ the smallest integer so that every $\mathcal{G}(n;F_k(n))$ contains an induced subgraph of degree k. How large is $F_k(n)$? Again it is trivial that $F_2(n)=n$. I observed that $F_3(n)<cn^{\frac{5}{3}}$ since it is easy to see that every $\mathcal{G}(n;cn^{\frac{5}{3}})$ contains either a $K(4)$ or an induced $K(3,3)$. It would be nice to improve this if possible.

Reference

[1] L. Pyber, Regular subgraphs of dense graphs, Combinatorica 5 (1985) 347–349.

7. Now I discuss some problems on extremal graph theory. Let G be any graph, denote by $T(n; G)$ the Turán number of G, i.e. the smallest integer so that every $\mathcal{G}(n; T(n; G))$ contains G as a subgraph. Many papers on this subject have been published recently (some by myself). Bollobás published an excellent book on this subject and Simonovits an excellent survey paper; thus to avoid repetitions I will try to mention as much as possible only new problems or questions which have been neglected.

In a paper Simonovits and I investigated the following question: Let $\mathcal{G}(n; T(n; G) + \ell)$ be a graph. How many copies of G must our graph contain? For large ℓ we get very satisfactory results but we had little success for small values of ℓ. In particular for which graphs G it is true that every $\mathcal{G}(n; T(n; \mathcal{G}))$ contains two (or more generally) many copies of G. If $\mathcal{G}$ is a triangle then Rademacher, almost immediately after he heard of the result of Turán $(T(n; K(3)) = [\frac{1}{4}n^2] + 1)$, proved that every $G(n; [\frac{1}{4}n^2] + 1)$ contains at least $[\frac{1}{2}n]$ triangles and this result is best possible. This result was extended to $G(n; [\frac{1}{4}n^2] + \ell)$ first for small values of g by me and later to a much larger range by Bollobás, Lovász and Simonovits. On the other hand Simonovits and I conjectured that $G(n; T(n; C_4))$ contains $c_1 n^{\frac{1}{2}}$ C_4's. If true this result is best possible, but we could not even prove that it contains 2 C_4's. Can one characterise those graphs G for which every $G_1(n; T(n; G))$ must necessarily contain at least two subgraphs isomorphic to G?

Here I would like to insert one problem on hypergraphs which perhaps will lead to interesting problems. I only state the simplest case. A classical problem of Turán states: Let $T^{(3)}(n; K^{(3)}(4))$ be the smallest integer so that every triple system on n elements and $T^{(3)}(n; K^{(3)}(4))$ triples contains a $K^{(3)}(4)$, i.e. a set of 4 elements all whose triples are in our system. The determination of $T^{(3)}(n; K^{(3)}(4))$ seems to be very difficult. Is it true that such a triple system must contain at least two (and perhaps in fact cn) $K^{(3)}(4)$'s. Observe that it is easy to see that every $G(n; [\frac{1}{4}n^2] + 1)$ contains an edge e and $c_1 n$ other vertices $X_1, \ldots, X_t$ so that all these vertices form a triangle with e in our $\mathcal{G}(n; [\frac{1}{4}n^2] + 1)$. Bollobás and I conjectured and Edwards proved that $c_1 = \frac{1}{6}$ is best possible. Is it true that every 3-uniform hypergraph (or triple system) $G^{(3)}(n; T^{(3)}(n; K^{(3)}(4))$ contains an edge e and $c_1 n$ vertices $x_1, \ldots, x_t$ so that e and x_i are a $K^{(3)}(4)$ in our hypergraph? This problem is of course open even for $t = 2$. It follows easily from our results with Simonovits that a $G^{(3)}(n; (1 + \varepsilon)T(n; K^{(3)}(4)))$ contains such a system.

References

[1] B. Bollobás, Relations between sets of complete subgraphs, Fifth British Combinatorial Conference, Utilitas Mathematicae (Winnipeg, 1976) 161–170.

[2] P. Erdős and M. Simonovits, Supersaturated graphs and hypergraphs, Combinatorica 3 (1983) 181–192.
[3] L. Lovász and M. Simonovits, On the number of complete subgraphs of a graph, Fifth British Combinatorial Conference, Utilitas Mathematicae (Winnipeg, 1976) 431–441.

8. Now I return to ordinary graphs, i.e. ($r=2$). The results of Stone, Simonovits and myself show that the most interesting open problems are if G is bipartite. I first state some of our favourite conjectures with Simonovits.

Is it true that for every bipartite G there is a rational $\alpha_1=\alpha(G_1)\leqslant\alpha<2$ for which

$$\lim_{n\to\infty} T(n;G)/n^{\alpha}=c(G), \quad 0<c(G)<\infty \tag{1}$$

exists? Is it true that for every rational α, $1\leqslant\alpha\leqslant 2$ there is a bipartite G for which the limit (1) exists? At present these conjectures which are about 20 years old are beyond our reach, and in fact we have no real evidence for their truth. We have no idea of the possible value of $c(G)$, it would perhaps be reasonable to assume that $c(G)$ is algebraic.

Let G be bipartite. Denote by $K(G)=r$ the largest integer for which G has an induced subgraph G' each vertex of which has degree $\geqslant r$. We conjectured that then

$$2-\frac{1}{r-1}<\alpha(G)\leqslant 2-\frac{1}{r}. \tag{2}$$

Thus in particular if $r=2$, i.e. if G has no induced subgraph each vertex of which has degree >2, then

$$T(n;G)<cn^{\frac{3}{2}}. \tag{3}$$

On the other hand if G has an induced subgraph each vertex of which has degree $\geqslant 3$ then our conjecture would imply $\alpha(G)>\frac{3}{2}$. There is some (admittedly inadequate) evidence for our conjectures. Let G be the graph defined by the edges of the three-dimensional cube. We proved

$$T(n;G)<cn^{\frac{8}{5}} \tag{4}$$

and we believe $\alpha(\mathcal{G})=\frac{8}{5}$, but unfortunately we could not even prove $\alpha(\mathcal{G})>\frac{3}{2}$. In fact even $T(n;G)/n^{\frac{3}{2}}\to\infty$ is open. We proved that $T(n;G-e)<cn^{\frac{3}{2}}$. Further we have

$$T(n;K(r,r)-e)<cn^{2-1/(r-1)}.$$

A nice test case of our conjecture is the following: Let G_t be a graph of $1+t+\binom{t}{2}$ vertices and $t+t(t-1)$ edges defined as follows:

The vertices of G_t are x; $y_1, y_2, \ldots, y_t$, $z_1, \ldots, z_{\binom{t}{2}}$. x is joined to all the y's and each z is joined to two of the y's so that every pair (y_i, y_j) is joined to exactly one z. Our conjecture (3) would imply

$$T(n;G_t)<c_t n^{\frac{3}{2}}. \tag{5}$$

For $t=3$ (5) is easy and we have no proof for $t \geq 4$. Omit X from G_t then we obtain G'_t. G'_3 is C_6 and we know $\alpha(G'_3) \approx \frac{4}{3}$. Faudree and Simonovits proved $\alpha(\mathcal{G}'_4) < \frac{3}{2}$. Perhaps for every t $\alpha(\mathcal{G}'_t) < \frac{3}{2}$. Unfortunately their ingenious proof only works for $t=4$.

Let now G_i, $1 \leq i \leq r$ be a family of graphs. Define $T(n; \mathcal{G}_1, \ldots, \mathcal{G}_r)$ as the smallest integer for which every $\mathcal{G}(n; T_n(\mathcal{G}_1, \ldots, \mathcal{G}_r))$ contains one of the G'_i's as a subgraph. Simonovits and I asked if there is a system $G_1, \ldots, G_r$ for which

$$\lim_{n\to\infty} T(n; G_1, \ldots, G_r) / \min_{i=1,2,\ldots,r} T_n(\mathcal{G}_i) = 0? \tag{6}$$

Perhaps our conjecture (1) can be generalised and there is an $\alpha(\mathcal{G}_1, \ldots, \mathcal{G}_r)$ for which

$$\lim_{n\to\infty} T(n; \mathcal{G}_1, \ldots, \mathcal{G}_r)/n^{\alpha(\mathcal{G}_1,\ldots,\mathcal{G}_r)} = c, \quad 0 < c < \infty \tag{7}$$

Faudree and Simonovits believe that

$$\alpha(C_4, \mathcal{G}'_4) < \min(\alpha(C_4), \alpha(\mathcal{G}'_4)) \tag{8}$$

and they hope that their method will give (8).

To end this long chapter I make a few remarks on some questions which certainly have not been investigated carefully. Let ℓ_k be the largest integer for which there is a $\mathcal{G}(k; \ell_k)$ satisfying

$$\lim_{n=} T(n; \mathcal{G}(k; \ell_k)/n^{\frac{3}{2}} < \infty. \tag{9}$$

$K(2; k-2)$ shows that $\ell_k \geq 2k-4$ and our conjecture (3) easily gives $\ell_k = 2k-4$. Perhaps this can be proved without (3) but by the probability method I could only prove that $T(n; G(k; 2k-3)) > cn^{\frac{3}{2}}$. Perhaps the following problem is more interesting: What is the largest integer t_k for which there is a $G(k; t_k)$ satisfying

$$T(n; \mathcal{G}(k; t_k)) = o(n^{\frac{3}{2}})? \tag{10}$$

If (10) holds then $\mathcal{G}(k; t)$ can of course not contain a C_4. It would be interesting to determine other forbidden bipartite graphs whose presence prevents (10) from holding. I have no good guess about the size of t_k, perhaps

$$2k - c_1 k^{\frac{1}{2}} < t_k < 2k - c_2 k^{\frac{1}{2}}. \tag{11}$$

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[3] M. Simonovits, Extremal graph problems, degenerate extremal problems, and supersaturated

graphs, Progress in graph theory, Silver Jubilee Conference Combinatorics (Univ. of Waterloo Acad. Press 1984) 919–937.
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9. Now I give a very short discussion of extremal problems on hypergraphs. To make the paper short I state only a few new problems. A long paper of mine is in the press on similarities and differences of extremal problems between graphs and hypergraphs and Frankl and Füredi have a long forthcoming paper in the J. Combin. Theory (A) on this subject.

Let $G_1^{(3)}$ be two triangles with a common edge and $G_2^{(3)}$ be the following $G^{(3)}(6;3)$ having the vertices x_1, x_2, x_3, x_4, x_5, x_6 and the edges $\{x_1, x_2, x_3\}\{x_3, x_4, x_5\}\{x_2, x_4, x_6\}$. An old problem of W. Brown, V.T. Sós and myself asked for the determination or estimation of $T^{(3)}(n; G_1^{(3)}, G_2^{(3)})$. Ruzsa and Szemerédi proved that

$$T^{(3)}(n; G_1^{(3)}, G_2^{(3)})/n^2 \to 0 \tag{1}$$

but for every $\varepsilon > 0$ $T^{(3)}(n; G_1^{(3)}, G_2^{(3)})/n^{2-\varepsilon} \to \infty$. In fact they prove a sharper result. An asymptotic formula for $T^{(3)}(n; G_1^{(3)}, G_2^{(3)})$ seems hopeless at present. (1) is certainly a new phenomenon. I then asked if there exists for some r a $G^{(r)}$ so that there is an α for which

$$T(n; G^{(r)})/n^{\alpha} \to 0$$

but for every $\varepsilon > 0$ $T(n; G^{(r)})/n^{\alpha-\varepsilon} \to \infty$. Frankl and Füredi found such a $G^{(r)}$ for $r = 5$, $\alpha = 4$. It is not yet known if for $r < 5$ such a hypergraph exists.

Frankl gave a talk on hypergraphs at our meeting in Hakone. After his lecture I asked: Is it true that every $G^{(3)}(n; 1 + \binom{n+1}{2})$ contains our $G_2^{(3)}$ and that the only $G^{(3)}(n; \binom{n-1}{2})$ which does not contain our $G_2^{(3)}$ consists of the $\binom{n-1}{2}$ triples which have a common vertex? Frankl informed me that this has already been proved by him and Füredi but that the proof is not quite simple. I at first though that if every vertex is contained in only $o(n^2)$ triples then every such $G^{(3)}(n; \varepsilon n^2)$ will contain our $G_2^{(3)}$. This was easily disproved by Frankl but perhaps such a $G^3(n)$ can only have $\frac{1}{2}n^2(1 + o(1))$ edges. I then asked: Is it true that if every pair of vertices (x, y) is contained in only $< Cn^{\frac{1}{2}}$ triples then every $G^{(3)}(n; \varepsilon n^2)$ must contain our $G_2^{(3)}$? During our excursion Füredi found the following nice counterexample:

Let $n = p^2 + p + 1$. We will have $2n$ elements A and L where A corresponds to the points and L to the lines of a finite geometry of n points. We divide A into two disjoint sets $B \cup C$, both having $(\frac{1}{2} + o(1))p^2$ elements and both meet every line of our finite geometry in $(\frac{1}{2} + o(1))\frac{1}{2}n^{\frac{1}{2}}$ points, Füredi's system now consists of the $(1 + o(1))\frac{1}{4}n^2$ triples (x, y, ℓ) where $x \in B$, $y \in C$ and $\ell \in L$ where x and y are on the line ℓ.

Whereupon I modified my conjecture:

Assume that every (x, y) is contained in only $o(n^{\frac{1}{2}})$ triples of our system. Then if such a triple system has cn^2 edges must it contain a $G_2^{(3)}$? It seems to speak against this conjecture that it was born as a response to several counterexamples. To end this section I would like to state an old problem of mine which seems very difficult: Is it true that for every k and $\varepsilon > 0$ there is an n_0 so that for every $n > n_0$ every $G^{(3)}(n; \varepsilon n^2)$ contains either a $G_2^{(3)}$ or a $G^{(3)}(k; k+3)$? For $k = 3$ this was our problem with Brown and V.T. Sós which was settled by Ruzsa and Szemerédi but for $k > 3$ very serious new difficulties appear and the problem is still very much open.

By the way Frankl proved a result on hypergraphs which is related to conjecture (1) of the previous chapter.

Reference

[1] P. Frankl, All rationals occur as exponents, J. Combin. Theory (A) 42 (1986) 200–206.

10. To end the paper I state a few miscellaneous problems. First of all here is a very nice problem of Tuza. Let $\mathscr{G}$ be a graph and k the largest integer for which G has k edge disjoint triangles. Is it then true that G can be made triangle free by the omission of at most $2k$ edges? $K(4)$ and $K(5)$ shows that if true the result is best possible. If true many generalisations and extensions will be possible.

Rothschild and I posed a few years ago the following problem: Assume that every edge of a $\mathscr{G}(n; cn^2)$ is contained in a triangle. Denote by $h(n; c)$ the largest integer so that every such graph has an edge which is contained in at least $h(n; c)$ triangles. The determination or good estimation of $h(n; c)$ does not seem to be quite easy. Szemerédi observed that his Regularity Lemma easily gives that for every $c > 0$

$$\lim_{n \to \infty} h(n; c) = \infty$$

and Noga Alon proved that for small c, $h(n; c) < c' n^{\frac{1}{2}}$. It is easy to see and well known that for $c > \frac{1}{4}$ one has $h(n; c) > c_1 n_1$, but without much difficulty the following stronger result can be proved.

Let $e > \frac{1}{4}n^2 - cn$ and assume that every edge of $G(n; e)$ is contained in a triangle. Then there is an absolute constant $c_1 = c_1(c)$ for which our $\mathscr{G}(n; e)$ has an edge which is contained in $\geqslant c_1 n$ triangles. That the result is best possible is a slight modification of Noga Alon's proof that if $f(n) \to \infty$ then there is a $G(n; (\frac{1}{4}n^2 - nf(n)))$ each edge of which is in a triangle but every edge is only in $o(n)$ triangles.

We give an outline of the proof. Let $G(n; \frac{1}{4}n^2 - cn)$ be a graph each edge of which is contained in a triangle. Observe first that if (x_1, x_2, x_3) is a triangle of our G, then we can assume that

$$v(x_1)+v(x_2)+v(x_3)\leqslant n(1+\varepsilon) \tag{1}$$

For if (1) would not hold then one of the edges (x_1, x_2), (x_1, x_3), (x_2, x_3) were contained in at least $\frac{1}{3}(\varepsilon n)$ triangles and thus our theorem is proved in this case. Henceforth we can assume that (1) holds for every triangle of our graph.

Assume next that our $G(n; \frac{1}{4}n^2 - cn)$ contains more than $10c$ vertex disjoint triangles $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$. Omit all these vertices from our G. Then by (1) we obtain a $\mathcal{G}(n-30c)$ which has more that $\frac{1}{4}(n-30c)^2$ edges and therefore by an old and elementary result of mine it contains an edge which is contained in $c'n$ triangles, by the result of Edwards $c' = \frac{1}{6} + o(1)$.

Thus we can assume that (1) holds and our G has at most $10c$ vertex disjoint triangles. But then by a simple argument at least one vertex is contained in $n^2/130c$ triangles and therefore at least one edge is contained in at least $n/130c$ triangles, which proves the first part of our theorem.

The proof of the second half of our assertion is very simple. Let $|G| = n$, $|A| = \ell_n^2$ where ℓ_n tends to infinity as slowly as we please. $|B| = |C| = \frac{1}{2}(n - \ell_n^2)$. Join every vertex of B to every vertex of C. Divide B and C into ℓ_n roughly equal disjoint sets B_i and C_j. Join $x_{i,j} \in A$ to every vertex of B_i and C_j. If ℓ_n tends to infinity sufficiently slowly our graph has $\frac{1}{4}(n^2) - nf(n)$ edges and each edge is on $o(n)$ triangles as stated.

It would perhaps be of interest to improve the estimates for $h(n; c)$ and investigate what happens if $c = c_n \to 0$.

Pyber and I considered the following related problem. Assume again that every edge of a $\mathcal{G}(n; e)$ is contained in at least one triangle. Denote by $L(n; e)$ the largest integer so that our graph contains at least $L(n; e)$ triangles. Trivially for all e $L(n; e) \geqslant e/3$, and the result of Ruzsa and Szemerédi shows that for $e < cnr_3(n)$ $e/3$ is exact. On the other hand it is well known and easy to see that if $e > (1+\varepsilon)\frac{1}{4}n^2$ then $L(n; e) > c_\varepsilon n^3$ even if we do not assume that every edge is contained in a triangle. We thought that perhaps for $e > cn^2$ $L(n, e) > (\frac{1}{2} + o(1))e$. The complete bipartite graph with a matching shows that if true this is best possible[1]. It would perhaps be of interest to investigate what happens to $L(n; e)$ if $e/n^2 \to 0$ very slowly.

Last year Stephan Burr and I came across the following problem: Let $f(n)$ be the smallest integer for which if we color the edges of $K(f(n))$ (i.e. a complete graph of $f(n)$ vertices) by two colors then there either are two monochromatic $K(n-1)$'s with a common vertex where the two $K(n-1)$'s have different colors or there is a monochromatic $K(n)$. Is it true that $f(n) = r(n; n-1)$? This is open even for $n = 5$.

The following simple problem of Renu Laskar and myself seems still to be

[1] P. Frankl has just proved this conjecture.

open. Let $g(n)$ be the largest integer for which every $\mathcal{G}(n;[\frac{1}{4}n^2]+1)$ contains a triangle x_1, x_2, x_3 for which the sum of the degrees of the vertices x_1, x_2, x_3 is $\geqslant g(n)$. We proved

$$(1+c)n < g(n) < (1+o(1))2(\sqrt{3}-1)n.$$

The upper bound is probably best possible. Clearly many generalisations and extensions are possible.