Commentary

Schoenberg's first paper, "Über die asymptotische Verteilung reeller Zahlen *mod* 1" [**1***], was an important paper. It presents a general theory of non-uniform distributions of numbers in [0,1]. The main result is that $\varphi(n)/n$ has a continuous distribution function, i.e., for each $z \in [0,1]$, the limit

$$\lim_{n\to\infty} \#\{m \le n : \varphi(m) \le zm\}/n$$

exists and depends continuously on z. Here, $\varphi(n)$ is Euler's totient function, i.e., $\varphi(n)$ gives the number of nonnegative integers less than n and relatively prime to it. Schoenberg used methods of Fourier analysis. Subsequent authors have given elementary proofs and many generalizations; the subject matter eventually developed into probabilistic number theory [**E**]. Very soon after [**1***] appeared, Behrend, Chowla [**C**], and Davenport [**D**] each proved independently that the density of abundant numbers exists by showing that $\sigma(n)/n$ has a continuous distribution function. Here, $\sigma(n)$ is the sum of the divisors of n, and n is called 'abundant' if $\sigma(n) > 2n$. A little later I proved this in [**Er1**] by elementary methods by showing that the sum of the reciprocals of the primitive abundant numbers converges. For further references and details, see Elliott's book [**E**].

Stimulated by [**D**] and other papers, Schoenberg returned to this subject in his paper [**18***], "On Asymptotic Distributions of Arithmetical Functions", in which he proved that, under fairly general conditions, the distribution function of a multiplicative function exists. This result was ultimately subsumed in the Erdős-Wintner theorem which gives a necessary and sufficient condition for the existence and continuity of the distribution function of an additive arithmetical function; see [**E**], especially Chapter 5. Schoenberg also gave a sufficient condition for the distribution function to be purely singular, and a sufficient condition for it to be continuous. I later proved (in [**Er2**]) that the negation of Schoenberg's sufficient condition for the singularity of the distribution function is necessary and sufficient for the continuity of the distribution function.

Schoenberg also asks for a necessary and sufficient condition for the absolute continuity of the distribution function. This problem is still unsolved and seems very difficult. A few years later I proved (in [**Er3**]) that the distribution function of $\sigma(n)/n$, and in fact of most of the usual arithmetic functions, is purely singular, but I gave examples of arithmetic functions whose distribution function is absolutely continuous and in fact is an entire function. Some of my conjectures were settled by G.J. Babu (see, e.g., [**B**]), but a necessary and sufficient condition for the absolute continuity of the distribution function is nowhere in sight and perhaps there is no simple condition.

Schoenberg's paper [**79**] on "Arithmetic problems concerning Cauchy's functional equation" is a brief report on the material in his joint paper [**81**] with Ch. Pisot with the same title. The authors consider the following problem. Let P be a set of k distinct primes and let A be the set of all integers composed of the p's in P. Assume that the function f is strictly monotone on A and satisfies

$$f(a) = \sum_{p^\alpha \| a} f(p^\alpha),$$

where $p^\alpha \| a$ means that p^α divides a but $p^{\alpha+1}$ does not. Does it then follow that $f(n) = c \log n$? Yes if $k > 2$ but No if $k = 2$. The authors also ask this question when the strict monotonicity of f is replaced by the condition that

$$f(a_{i+1}) - f(a_i) \rightarrow 0, \tag{1}$$

where $a_1 < a_2 < \dots$ gives the elements of A in order. As far as I know this problem is still open. I conjectured and Wissing proved that if $f(n+1) - f(n) < c$, then $f(n) = \alpha \log n + g(n)$ for some bounded g. Perhaps (1) could be replaced by

$$f(a_{i+1}) - f(a_i) < c \tag{1$'$}$$

and this might imply that $f(n) = c \log n + g(n)$ for some bounded g.

References

[B] G.J. Babu, Absolutely continuous distribution functions of additive functions $f(p) = (\log p)^{-a}, a > 0$, *Acta Arith.* **26** (1974/5), 401–403.

[C] S. Chowla, On abundant numbers, *J. Indian Math. Soc.* **2** (1934), 30–34.

[D] H. Davenport, Über numeri abundantes, *Sitzungsber. Preuss. Akad. Wiss. Berlin* **27** (1933), 830–837.

[E] P.D.T.A. Elliott, *Probabilistic Number Theory*, Vol. 1 and 2, Springer-Verlag, 1980.

[Er1] P. Erdős, On the density of abundant numbers, *J. London Math. Soc.* **9** (1934), 278–282.

[Er2] P. Erdős, On the density of some sequence of numbers. III, *J. London Math. Soc.* **13** (1938), 119–127.

[Er3] P. Erdős, On the smoothness of the asymptotic distribution of additive arithmetical functions, *Amer. J. Math.* **61** (1939), 722–725.

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